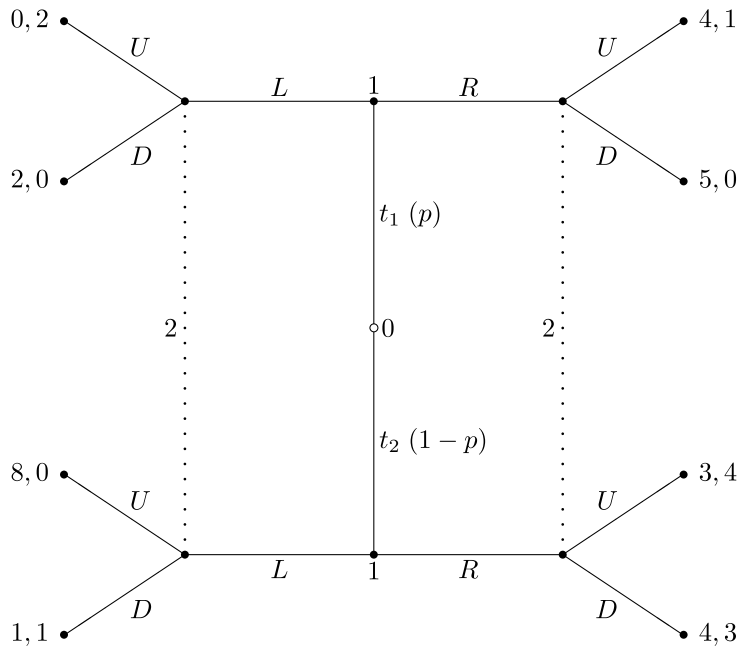
**Microeconomic Theory II** **Spring 2016**

**Final Exam Solutions** **Mikhael Shor**

Warning: Brief, incomplete, and quite possibly incorrect.

**Question 1.** Consider the following game. First, nature (player 0) selects $t_1$ with probability $p$, $0 < p < 1$, or $t_2$ with probability $1-p$. Next, player 1 selects $L$ or $R$. Lastly, player 2 selects $U$ or $D$.




(a) Find all pure-strategy weak Perfect Bayesian Nash equilibria of this game. Carefully explain.

First, note that $t_1$ would never select $L$ since the highest possible payoff (2) is lower than the lowest possible payoff from $R$ (4). Thus, we need only consider one separating and one pooling equilibrium.

Consider the separating equilibrium in which the sender chooses:

$$t_1 \rightarrow R, \;\; t_2 \rightarrow L$$

The beliefs are degenerate, and the receiver's unique best response is:

$$R \rightarrow U, \;\; L \rightarrow D$$

However, $t_2$ then earns 1 from the presumed equilibrium strategy of $L$, but earns 3 from $R$. So, this is not an equilibrium.

Next, consider the pooling equilibrium where both types select $R$. The receiver's on-equilibrium beliefs are $\mu(t_1|R) = p$ and best response to $R$ is $U$ for all $p$ (because $1 > 0$ and $4 > 3$). To check sender's best replies, note again that $t_1$ always wants to play $R$. However, $t_2$ must prefer his equilibrium payoff from $R$ to what he can earn from $L$. This requires the receiver's strategy at $L$ to be $D$, and this requires $\mu(t_1|L) \leq \frac{1}{3}$. Therefore, the only pure-strategy Perfect Bayesian Nash equilibrium is:

$$t_1 \to R, \;\; t_2 \to R$$

$$\mu(t_1|L) = q, \;\; \mu(t_1|R) = p$$

$$R \to U, \;\; L \to D$$

for any $q \leq \frac{1}{3}$.

(b) Which of the equilibria above satisfy the intuitive criterion? Carefully explain.

- The only unsent message is $L$.
- Type $t_1$ is earning 4 in equilibrium and would never choose $L$ no matter what player 2 could reasonably do.
- Imagine $t_2$ chooses $L$. The receiver knows that this is not $t_1$ (and therefore must be $t_2$). The receiver would select $D$. But then $t_2$ would *not* defect. Thus, the equilibrium *does* satisfy the intuitive criterion.

(c) How does your answer above in part (a) depend on $p$? Carefully explain why this is the case.

It does not. This is because, in a pooling equilibrium, $p$ is important only for on-equilibrium beliefs (see equilibrium beliefs in (a)) which *may* impact the best response of the receiver. Here, however, in response to $R$, $U$ is always strictly better than $D$ for any $p$.

(d) Briefly discuss how and when, *in general*, the existence of some Perfect Bayesian Nash equilibria in signaling games may depend on $p$.

It has no effect on the existence or nature of separating equilibria, because the on-equilibrium beliefs are always degenerate (0 or 1). For the existence of pooling equilibria, $p$ may affect the receiver's on-equilibrium best responses, which may affect existence.

**Question 2.** Consider a principal-agent model in which the agent chooses between two levels of effort, $\{e_l, e_h\} = \{0, 1\}$. The principal pays the agent a wage $w_s$ in state $s$ and realizes output of $\pi_s$. There are four states, with the probability of a state contingent on effort given by:

| effort level | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ |
|---|---|---|---|---|
| $e_l(=0)$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ | $\frac{1}{8}$ |
| $e_h(=1)$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |

The agent's utility function is $\sqrt{w} - e$ and his reservation utility is $\underline{u} = 2$. The principal is risk neutral, with utility given by $\pi - w$.

(a) Compute the wage schedule that optimally implements $e_h$ when effort is observable.

When effort is observable, wages may depend on both the state and effort. However, risk aversion implies that the optimal wage must not vary with state. Thus, the optimal wage satisfies IR with equality:

$$\sqrt{w_h} - e = \underline{u} \Rightarrow w_h = 9.$$

Implied in the above is that $w_l$ is some sufficiently low wage that yields nonpositive utility for the agent (e.g., $w_l = -1,000,000$).

(b) Compute the wage schedule that optimally implements $e_h$ when effort is unobservable.

The constraints are:

$$IC: \frac{1}{4}\sqrt{w_1} + \frac{1}{4}\sqrt{w_2} + \frac{1}{4}\sqrt{w_3} + \frac{1}{4}\sqrt{w_4} - 1 \geq \frac{3}{8}\sqrt{w_1} + \frac{3}{8}\sqrt{w_2} + \frac{1}{8}\sqrt{w_3} + \frac{1}{8}\sqrt{w_4}$$

$$IR: \frac{1}{4}\sqrt{w_1} + \frac{1}{4}\sqrt{w_2} + \frac{1}{4}\sqrt{w_3} + \frac{1}{4}\sqrt{w_4} - 1 \geq 2$$

The most important thing to note is that states 1 and 2 must result in the same wages, and states 3 and 4 must result in the same wages. This is because the *ratio* of the probabilities is the same! Note that this does not require the probabilities to be the same (this implies that the ratio is the same, but is not necessary for wages to be equal) and it is not because the difference in the probabilities are the same. From the FOCs of the general derivation, we found that wages depend only on the ratio of probabilities. Given the above, the IC and IR constraints reduce to:

$$IC: \sqrt{w_3} - \sqrt{w_1} \geq 4$$

$$IR: \sqrt{w_1} + \sqrt{w_3} \geq 6$$

As both of these constraints must bind, the solution is

$$w_1 = w_2 = 1, \;\; w_3 = w_4 = 25$$

(c) Consider the wage schedule $\{w_1, w_2, w_3, w_4\} = \{1, 1, 16, 36\}$. Does this wage schedule implement $e_h$? Does it *optimally* implement $e_h$? Explain.

By plugging in the wages into the IC and IR constraints above, we see that both constraints are satisfied. Therefore, the wage schedule *does* implement $e_h$. To see that it does not do so optimally, we note that (i) the FOCs are not satisfied (as they imply that wages in states 1 and 2, and in 3 and 4 must be the same). Alternately, we can see that the expected wages in our solution above ($\frac{1}{4}(1+1+25+25) = 13$) are lower than in this proposed solution ($\frac{1}{4}(1+1+16+36) = 13.5$).

**Question 3.** Consider a game consisting of two repetitions of the following stage game:

| | | Player 2 | | |
|---|---|---|---|---|
| | | $A$ | $B$ | $C$ |
| Player 1 | $X$ | $6,8$ | $0,9$ | $1,3$ |
| | $Y$ | $2,0$ | $3,3$ | $2,2$ |
| | $Z$ | $6,2$ | $2,1$ | $5,4$ |

Players observe the outcome of the first stage before playing the second, with payoffs consisting of the sum of the two stages.

(a) Find the pure-strategy subgame-perfect Nash equilibrium that results in the *lowest* total payoff for the two players. Find the pure-strategy subgame-perfect Nash equilibrium that results in the *highest* total payoff for the two players.

The lowest payoff is obtained by playing $\{Y, B\}$ in every subgame. For the highest, note that there are two pure-strategy Nash equilibria of the stage game and thus only two candidates for play in the second stage: $\{Y, B\}$ and $\{Z, C\}$. However, we can do better in the first stage. Consider the following equilibrium:

Player 1: Play $X$ in the first period, play $Z$ in the second period following $\{X, A\}$ and play $Y$ in the second period following any other outcome.

Player 2: Play $A$ in the first period, play $C$ in the second period following $\{X, A\}$ and play $B$ in the second period following any other outcome.

The above specifies a Nash equilibrium in the second stage in every subgame and makes $\{X, A\}$ part of the SPNE in the first stage. To see this, add the second-period payoffs to the matrix above and observe that $\{X, A\}$ is an equilibrium.

(b) Suppose that the payoffs $(3,3)$ from $(Y, B)$ are replaced by $(4,3)$. How does this change your answers above? Would you expect player 1 to benefit from this increased payoff? Briefly discuss and explain.

This obviously improves the lowest possible equilibrium payoff but has no effect on the highest payoff as the above is still an equilibrium.

(c) Suppose that the payoffs $(3,3)$ from $(Y, B)$ are replaced by $(3,4)$. Would you expect player 2 to benefit from this increased payoff? Briefly discuss and explain.

This obviously improves the lowest possible equilibriu payoff but also eliminates the above highest-payoff equilibrium because $A$ would no longer be a best reponse in the first period! This is one of many examples in game theory where a higher payoff in some scenarios can actually hurt you in equilibrium.