Microeconomic Theory II — Spring 2020

Final Exam SOLUTIONS — Mikhael Shor

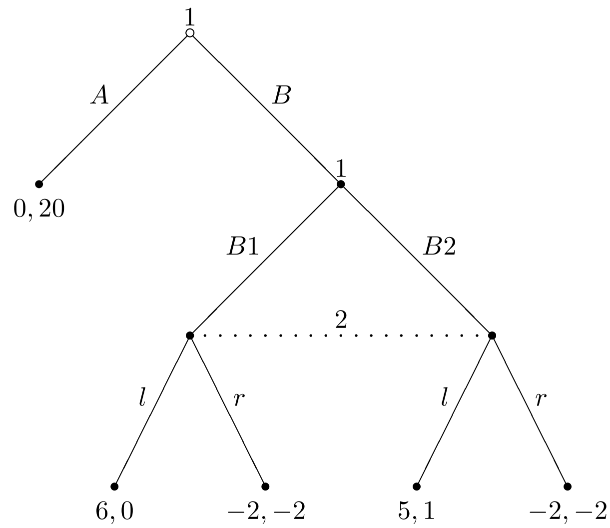
Carefully explain and support your answers.

Question 1. This question has you consider how Nash equilibrium outcomes differ from the outcomes of two Nash refinements: subgame-perfect Nash equilibrium and weak perfect Bayesian Nash equilibrium. For each of the four games drawn below, find all pure-strategy NE, SPNE, and wPBNE. For each game, discuss intuitively where the three equilibrium concepts coincide or differ and why. Then, briefly discuss generally what the four examples illustrate about the three solution concepts.

The overall goal of this question was to show two things: (1) Examples of games in which SPNE or PBE don't help refine NE and (2) games that collectively show that SPNE and PBE can't be nested. Sometimes SPNE is the more useful refinement, sometimes PBE is, and this is because they work in different ways.

As an aside, an equilibrium notion that refines *both* SPNE and PBE is the notion of *sequential equilibrium* which is like a trembling-hand version of PBE with small mistakes that ensure that every information set is reached with positive probability.

Game 1:



The pure strategy NE are $\{A, B1; r\}, \{A, B2; r\}, \{B, B1; l\}$

In the only proper subgame, the unique pure-strategy equilibrium is $\{B1, l\}$. Therefore, the only pure strategy SPNE is $\{B, B1; l\}$

For any beliefs of player 2, only l is sequentially rational. Therefore, the only wPBE is $\{B, B1; l\}$ with $\mu(B1) = 1$.

Here, both SPNE and wPBE yield a proper subset of NE, but are identical in their prediction as both essentially just restrict player 2 to selecting l.

Remark 1.1a Be sure to include beliefs in your description of a wPBE, and make sure that the notation specifies where in the information set you are: e.g., $\mu(B1)$ or $\mu(left)$ (but $\mu(l)$ versus $\mu(r)$ does not help).

Remark 1.1b Note that "pure strategy" does not imply degenerate beliefs. That is, one can have a pure-strategy PBE with beliefs that do not assign probability 1 to one of the nodes in an information set. In fact, if a player has more than three actions at an information set, making one of them sequentially rational might *require* interior beliefs.

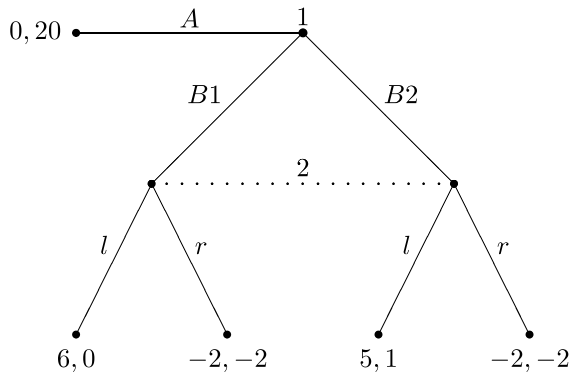
Remark 1.1c PBE is a refinement of NE (not of SPNE). Therefore, a NE that isn't SPNE should still be checked when identifying PBE.

Remark 1.1d Note that beliefs serve two purposes in PBE: they must be consistent with Bayes rule and they must allow the succeeding action to be sequentially rational. For $\{B, B1; l\}$, any beliefs make l sequentially rational, but only $\mu(B1) = 1$ is consistent with Bayes rule and thus part of a PBE.

Remark 1.1e Player 1 has four pure strategies, $\{A\ B1, A\ B2, B\ B1, B\ B2\}$. That is how this game differs from game 2. In particular, A is not a strategy for player 1 as it prescribes no action for player 1 in the proper subgame.

Remark 1.1f Be precise in your language. Player 2 would not play r no matter the beliefs at the information set. This means that r is never sequentially rational. It *does not* mean that r is strictly dominated (in response to any strategy for player 1 that includes A, player 2 is indifferent between l and r).

Game 2:



This is similar to the above, except player 1's two decisions are collapsed into one (and thus the proper subgame is gone).

The pure strategy NE are $\{A; r\}, \{B1; l\}$

There are no proper subgames. Therefore, SPNE and NE coincide. The pure strategy SPNE are $\{A; r\}, \{B1; l\}$

For any beliefs of player 2, only l is sequentially rational. Therefore, the only wPBE is $\{B1; l\}$ with $\mu(B1) = 1$.

Here, SPNE does not help while wPBE yields a proper subset of NE, again by restricting player 2 to selecting l. This is very similar to the example we used to introduce the idea of PBE as it forces a sequentially rational decision at player 2's information set while SPNE only does so if it is contained in a proper subgame as in Game 1.

Remark 1.2a Be careful to realize that there are no proper subgames in this game. Player 1 choosing between $B1$ and $B2$ does not begin a subgame (since all actions must be included, and that includes A). Therefore, SPNE and NE coincide.

Game 3:

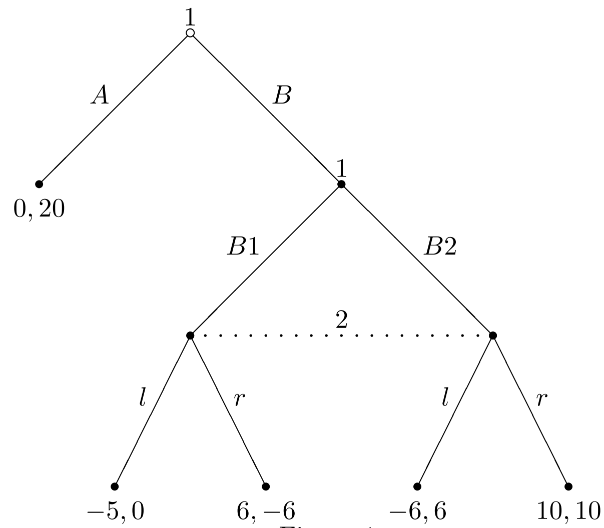


Figure 1:

The pure strategy NE are $\{A, B1; l\}, \{A, B2; l\}, \{B, B2; r\}$

In the only proper subgame, the pure-strategy NE are $\{B1, l\}$ and $\{B2, r\}$. Therefore, the pure strategy SPNE are $\{A, B1; l\}, \{B, B2; r\}$

For PBE, we have:
$\{A, B1; l\}$ with beliefs, e.g., $\mu(B1) = 1$, but generally $\mu(B1) \geq \frac{2}{5}$
$\{A, B2; l\}$ with beliefs, e.g., $\mu(B1) = 1$, but generally $\mu(B1) \geq \frac{2}{5}$
$\{B, B2; r\}$ with beliefs $\mu(B2) = 1$

Here, wPBE offers no refinement at all since off-equilibrium beliefs allow for any action by player 2 (and thus allow the strange-seeming beliefs in the second PBE above), while SPNE requires that the action by player 2 at least be a best response in the proper subgame.

Remark 1.3a When player 1 selects either $A, B1$ or $A, B2$, there are no restrictions on player 2's beliefs. Since player 2 has beliefs for which l is a best response, and since player 1's best response to l is to play $A, B1$ or $A, B2$, both NE with A and l are supported by a PBE.

Remark 1.3b This is a minor issue, but be careful in transcribing payoffs from the extensive form to the strategic form (for the whole game and for the subgame) and be careful underlining the best replies. Obviously, small transcription errors or errors in identifying best replies can alter the equilibrium analysis.

Remark 1.3c Note that the restriction on beliefs, $\mu(B1) \geq \frac{2}{5}$, uses a weak inequality and not a strict one. This is because when the player is indifferent, either action is a best reply.

Remark 1.3d It is important to check if every NE is potentially part of a PBE, not just check every SPNE.

Remark 1.3e Some students attempted to incorporate the idea of SPNE into PBE by arguing, for example, that the best response to player 2 playing l is to play $B1$. This is correct in the subgame, but not in the game as a whole where the best response to l is either $A, B1$ or $A, B2$. PBE does not have any notion of subgame in its definition so both $\{A, B1; l\}$ and $\{A, B2; l\}$ can be supported as PBE with appropriately chosen beliefs.

Remark 1.3f Since SPNE is a refinement of NE, it is not possible to find a SPNE that is not a NE. SPNE are NE that *also* induce a NE in every proper subgame (in this case, that just implies that they are NE that contain either $\{B1, l\}$ or $\{B2, r\}$).

Game 4:

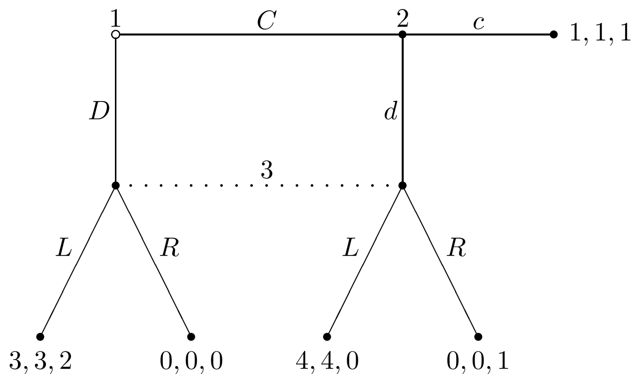


Figure 2:

This game is called "Selten's Horse" and shows the difference between (i) subgame perfection and (ii) sequential rationality at a singleton node (here, for player 2). It is somewhat tricky.

The pure strategy NE are $\{C; c; R\}, \{D; c; L\}$

Since there are no proper subgames, the pure strategy SPNE are $\{C; c; R\}, \{D; c; L\}$

For PBE, we check each NE above:

If $\{C; c; R\}$ then any beliefs are consistent for player 3. If player 3 were to play L, player 2 would deviate to d. Therefore, player 2 must play R, so $\mu(D) \leq \frac{1}{3}$.

If $\{D; c; L\}$ then $\mu(D) = 1$ and player 3 will play L. However, player 2 here must play a best response given his beliefs at his only information set, which is a singleton. Given player 3 will play L, player 2's best response given he reaches his information set is d. Therefore, this is not a PBE.

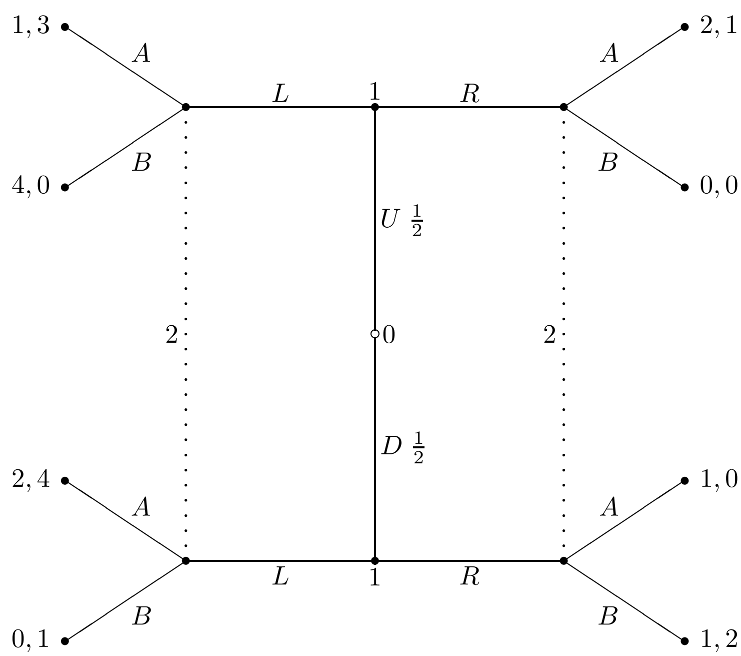
Therefore, there is a unique (pure strategy) PBE: $\{C; c; R\}$ with $\mu(D) \leq \frac{1}{3}$.

Remark 1.4a $\{D, d, L\}$ is not a NE as player 1 would deviate to C $(4 > 3)$.

Remark 1.4b Conceptually, PBE off the equilibrium belief allows for any probability distribution (beliefs) at each information set and any action that is a best response given that belief (sequential rationality). Here, if player 1 plays D, player 2 has an information set unreached in equilibrium. We can ascribe any beliefs that are a legitimate probability distribution (that is, only a probability of 1 to being at that node) and any best response to them (given player 3 is playing L, that must mean d). This is no different than what we are doing for player 3, except the information set is degenerate.

Remark 1.4c This was probably a typo/misstatement but be careful to note that this game having no proper subgames implies that SPNE are the same as NE, not that no SPNE exist!

Question 2. Consider the following game. First, nature (player 0) selects U with probability $\frac{1}{2}$ or D with probability $\frac{1}{2}$. Next, player 1 selects L or R. Lastly, player 2 selects A or B.



(a) Find all pure-strategy weak perfect Bayesian equilibria.

We could go through all four possibilities, but first let's reduce the number of cases we need to consider. Note that, for any beliefs following L, player 2 will always select A. Further, since any PBE must have player 2 selecting A following L, type D will always pick L $(2 > 1)$. So we only need to consider two cases.

Case 1: separating
(1) Assume $U \to R, D \to L$
(2) $\mu(U|R) = \mu(D|L) = 1$
(3) $L \to A, R \to A$
(4) $U \to R \to A \to 2, U \to L \to A \to 1$, so that works
$D \to L \to A \to 2, D \to R \to A \to 1$, so that works
Therefore: (1), (2), and (3) above constitute a (separating) PBE
Specifically, $\{U \to R, D \to L; A, A\}$, $\mu(U|R) = \mu(D|L) = 1$

Case 2: pooling
(1) Assume $U \to L, D \to L$
(2) $\mu(U|L) = \frac{1}{2}, \mu(U|R) = ?$
(3) $L \to A, R \to ?$
(4) $U \to L \to A \to 1, U \to R \leq 1$ so we need $R \to B$
$D \to L \to A \to 2, D \to R \to (A \vee B) \to 1$
Therefore:
$\{U \to L, D \to L; L \to A, R \to B\}$
$\mu(U|L) = \frac{1}{2}, \mu(U|R) \leq \frac{2}{3}$
is a (pooling) PBE

Remark 2.1a The pooling equilibrium exists for any beliefs $\mu(U|R) \leq \frac{2}{3}$ with a *weak*, not strict, inequality since anything is a best response when all actions yield identical payoffs.

Remark 2.1b Remember that the beliefs are part of the definition of any type of Bayes equilibrium, and not just an intermediate step in its calculations. That is, when writing "the equilibrium is..." the beliefs should be included along with the strategies.

Remark 2.1c Remember that *both* players' strategies need to be specified in the equilibrium.

(b) How would your answer above change if the probability of U and D changed? Explain.

The separating equilibrium does not depend on the probabilities as equilibrium beliefs are always degenerate.

For the pooling equilibrium, it would usually depend on the probabilities but, in this case, A is the only sequentially rational choice for player 2 following L for any beliefs (and thus, for any prior probabilities) and (as above) the only possible equilibria must involve $D \to L$.

Remark A complete answer should specify that (i) separating equilibria would be unaffected generally, (ii) the pooling equilibrium with both types picking L is unaffected (except equilibrium path beliefs) because player 2's best response to L doesn't vary with beliefs here, but would be affected in general, and (iii) provide some discussion about why the other potential pooling equilibrium wouldn't become one regardless of probabilities (which is the case in this game because type D would never pick R for the reasons given above).

Question 3. Consider a differentiated-products version of a Bertrand duopoly. Firm $i \neq j$ has demand given by

$$q_i = 90 - 2p_i + p_j$$

with no costs of production. Each firm simultaneously selects p_i to maximize profit, $p_i q_i$.

1. Determine each firm's best response function.

   Maximizing profit (and omitting here a careful accounting of corner solutions and second order conditions),

   $$\frac{d\left[p_i q_i(p_i)\right]}{dp_i} = \frac{d\left[p_i\left(90 - 2p_i + p_j\right)\right]}{dp_i} = 0$$

   yields best response functions $p_i(p_j) = \frac{1}{4}\left(90 + p_j\right)$, $i, j \in \{1, 2\}, i \neq j$

   Prices are *strategic complements* in that an increase in price by one firm makes the other firm want to increase its price.

2. What is the Nash equilibrium? What are the resulting profits for each firm?

   Solving $p_1 = \frac{1}{4}\left(90 + p_2\right)$ and $p_2 = \frac{1}{4}\left(90 + p_1\right)$ yields $p_1 = p_2 = 30$ which is the Nash equilibrium.

   Profit is given by $p_i(90 - 2p_i + p_j) = 30(60) = 1800$

Now imagine that the firms compete by selecting quantities.

3. Find the inverse demand function, $p_i(q_i, q_j)$.

   $q_i = 90 - 2p_i + p_j \equiv p_i = 45 - \frac{1}{2}q_i + \frac{1}{2}p_j = 45 - \frac{1}{2}q_i + \frac{1}{2}\left(45 - \frac{1}{2}q_j + \frac{1}{2}p_i\right)$

   Thus $p_i = 90 - \frac{2}{3}q_i - \frac{1}{3}q_j$

4. If each firm simultaneously selects q_i, determine each firm's best response function.

   This is a differentiated Cournot model. Note that a firm's own quantity decision has a bigger impact on its price than the other firm's quantity decision. At the extreme level of differentiation, one firm's quantity would have no effect on the other firm's price.
   Maximizing $q_i p_i(q_i)$ with respect to q_i yields the best response function: $q_i(q_j) = \frac{3}{4}90 - \frac{1}{4}q_j$.

   Quantities are *strategic substitutes* as increasing one firm's quantity leads to a reduction in the other firm's quantity.

5. What is the Nash equilibrium? What are the resulting profits?

Solving $q_1 = \frac{3}{4}90 - \frac{1}{4}q_2$ and $q_2 = \frac{3}{4}90 - \frac{1}{4}q_1$ yields $q_1 = q_2 = 54$ which is the Nash equilibrium.

Profit is given by $q_i(90 - \frac{2}{3}q_i - \frac{1}{3}q_j) = 54(36) = 1944$

Therefore, quantity competition yields higher profits for firms than price competition here (a result we already confirmed in class when goods are not differentiated).