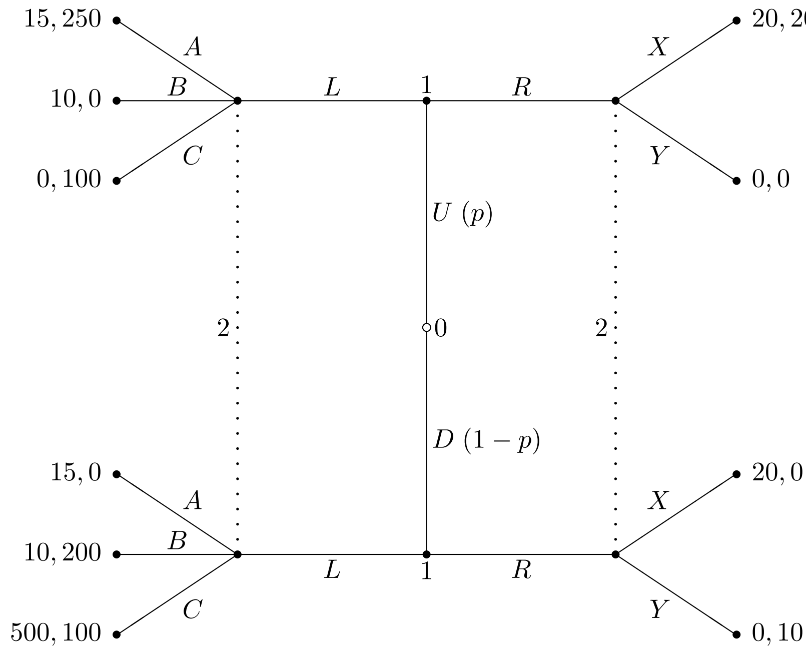
Microeconomic Theory II — Spring 2022

Final Exam SOLUTIONS — Mikhael Shor

Carefully explain and support your answers.

Question 1. Consider the following game. First, nature (player 0) selects U with probability p or D with probability $1-p$. Next, player 1 selects L or R. Lastly, player 2 selects either A, B, or C (if player 1 selected L) or X or Y (if player 1 selected R).



(a) What are each player's pure strategies?

For player 1: { LR, RL, RR, LL}
For player 2: { AX, BX, CX, AY, BY, CY }

Remark 1a.1 The sender (player 1) has two information sets–one for each type. {L,R} is the set of actions for *each* type. (Although in some cases, it may be convenient to think of each sender type as a separate player).

Remark 1a.2 The receiver (player 2) has two information sets. A strategy is not just an action taken at one of those information sets but a pair of actions, one for each information set.

(b) Assume $p = \frac{1}{2}$. Find all pure-strategy weak perfect Bayesian equilibria (and show that none other exist).

Separating equilibria:

1. $U \to L, D \to R$	1. $U \to R, D \to L$
2. (degenerate beliefs)	2. (degenerate beliefs)
3. $L \to A, R \to Y$	3. $R \to X, L \to B$
4. But D would defect $(15 > 0)$	4. But D would defect $(20 > 10)$

Therefore, there is no separating equilibrium.

Pooling on L:
1. $U \to L, D \to L$
2. $\mu(U|L) = \frac{1}{2}, \mu(U|R) \leq \frac{1}{3}$ (we require $R \to Y$ for equilibrium to exist.)
3. $L \to A, R \to Y$
4. Since $15 \geq 0$ neither player defects.

Therefore: $\{U \to L, D \to L; L \to A, R \to Y\}$ with beliefs $\mu(U|L) = \frac{1}{2}, \mu(U|R) \leq \frac{1}{3}$ is a wPBNE.

Pooling on R:
1. $U \to R, D \to R$
2. $\mu(U|R) = \frac{1}{2}, \mu(U|L) =?$
3. $R \to X, L \to A$ if $\mu(U|L) \geq \frac{4}{9}, L \to B$ if $\mu(U|L) \leq \frac{4}{9}$
4. Since $20 \geq 15$ and $20 \geq 0$ neither player defects.

Therefore:
$\{U \to R, D \to R; R \to X, L \to A\}$ with beliefs $\mu(U|R) = \frac{1}{2}, \mu(U|L) \leq \frac{4}{9}$
and
$\{U \to R, D \to R; R \to X, L \to B\}$ with beliefs $\mu(U|R) = \frac{1}{2}, \mu(U|L) \geq \frac{4}{9}$
are wPBNE

Remark 1b.1 Noting that a pooling equilibrium of some type exists in this game is insufficient. A Perfect Bayesian equilibrium requires a description of each player's strategies and also requires consistent beliefs on and off the equilibrium path for which those strategies are sequentially rational. This is especially important for the pooling equilibrium on R when the nature of the equilibrium (player 2's strategy) depends on the beliefs.

Remark 1b.2 {L,L} and {R,R} are not equilibria. They are the sender's (player 1's) strategies in pooling equilibria, but a PBNE must specify *each* player's strategies *and* beliefs.

Remark 1b.3 When determining the range of off-equilibrium beliefs for which an equilibrium exists, the range should use a weak inequality since indifference implies that any action is a best response. For example, a pooling equilibrium on L exists not only when $\mu(U|R) < \frac{1}{3}$ but also when $\mu(U|R) = \frac{1}{3}$ since it still allows $R \to Y$ to be part of an equilibrium.

Remark 1b.4 Note that C is never optimal for player 2 in response to L. For A to be optimal, we require :
$250\mu \geq 200(1-\mu)$ and $250\mu \geq 100 \Rightarrow \mu \geq \frac{4}{9}$ (where $\mu = \mu(U|L)$),
and for B to be optimal,
$250\mu \leq 200(1-\mu)$ and $200(1-\mu) \geq 100 \Rightarrow \mu \leq \frac{4}{9}$. For C to be optimal, we would need, $100 \geq 250\mu$ and $100 \geq 200(1-\mu)$ which is impossible.

(c) For each equilibrium found above, show whether or not it satisfies the Intuitive Criterion.

For pooling on L, no player earns strictly lower payoffs from deviating (both earn strictly more if they play R and player 2 responds with X). Therefore, it satisfies the intuitive criterion.

For pooling on R, neither player has an incentive to deviate for *any* off-equilibrium beliefs (note C will never be played). Thus, even if one player is identified as the one who would never deviate, the other would not then want to deviate. It satisfies the intuitive criterion.

Remark 1c.1 It is not enough to confirm that one type would want to deviate if its type is known by the receiver. You also must confirm that the other type would never deviate regardless of the beliefs of the receiver. For example, when pooling on L, the U type would want to deviate if its type is known (since receiver would play X), but the D type also has incentive to deviate for some beliefs. Since we can't identify a type who would never deviate, it satisfies the intuitive criterion.

(d) For what values of p does this game have a pooling equilibrium? Demonstrate or explain.

If sender pools on L, receiver will choose either A or B depending on p. In either case, there exist beliefs in response to R where the receiver chooses Y. Since both A and B earn more than Y for the sender of either type, this implies the existence of a pooling equilibrium for any p.

Question 2. Consider a principal-agent model in which the agent has three levels of effort, $e \in \{L, M, H\}$. There are three different outcomes associated with different profits for the principal, (π_1, π_2, π_3). Define p_i^e as the probability of outcome i when level of effort is e.

The principal is risk neutral with utility given by profits minus wages. The agents utility function is (of course) given by $u(w, e) = \sqrt{w} - c(e)$.

The cost to the agent of the three types of effort are $c(L) = 0, c(M) = 200, c(H) = 500$. Reservation utility is 0.

		outcome 1	outcome 2	outcome 3
(π_1, π_2, π_3)	$=$	$1,000,000$	$4,000,000$	$8,000,000$
(p_1^L, p_2^L, p_3^L)	$=$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
(p_1^M, p_2^M, p_3^M)	$=$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
(p_1^H, p_2^H, p_3^H)	$=$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

Wages cannot be negative.

(a) If effort can be observed, what is the optimal contract? Demonstrate.

$e = L: \quad \frac{1}{2}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{4}\pi_3 - c(L)^2 = 3,500,000 - 0 = 3,500,000$
$e = M: \quad \frac{1}{4}\pi_1 + \frac{1}{2}\pi_2 + \frac{1}{4}\pi_3 - c(M)^2 = 4,250,000 - 40,000 = 4,210,000$
$e = H: \quad \frac{1}{4}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{2}\pi_3 - c(H)^2 = 5,250,000 - 250,000 = 5,000,000$

Thus, the optimal compensation is $w(H) = 250,000$, along with sufficiently low payments for $w(L)$ and $w(M)$.

(b) Assume that effort cannot be observed (but outcomes can). Derive the optimal contract for *each* level of effort. Show all constraints.

For high and medium effort, the constraints simply require that the state most likely to occur under that effort level is sufficiently rewarded while all other states can be paid zero.

Start with the constraints for high effort (using $u_i = \sqrt{w_i}$):

$$
\begin{aligned}
\frac{1}{4}u_1 + \frac{1}{4}u_2 + \frac{1}{2}u_3 - 500 &\geq \frac{1}{2}u_1 + \frac{1}{4}u_2 + \frac{1}{4}u_3 - 0 \\
\equiv \frac{1}{4}(u_3 - u_1) &\geq 500 && IC1(H \succeq L) \\
\frac{1}{4}u_1 + \frac{1}{4}u_2 + \frac{1}{2}u_3 - 500 &\geq \frac{1}{4}u_1 + \frac{1}{2}u_2 + \frac{1}{4}u_3 - 200 \\
\equiv \frac{1}{4}(u_3 - u_2) &\geq 300 && IC2(H \succeq M) \\
\frac{1}{4}u_1 + \frac{1}{4}u_2 + \frac{1}{2}u_3 - 500 &\geq 0 && IR0 \\
u_1 &\geq 0 && IR1 \\
u_2 &\geq 0 && IR2 \\
u_3 &\geq 0 && IR3
\end{aligned}
$$

Note (i) $IR1$ and $IC1$ imply $IR3$; (ii) $IR1$, $IR2$ and $IC1$ imply $IR0$. Therefore, we are left with:

$$
\begin{aligned}
&u_3 \geq 2000 + u_1 && IC1(H \succeq L) \\
&u_3 \geq 1200 + u_2 && IC2(H \succeq M) \\
&u_1 \geq 0 && IR1 \\
&u_2 \geq 0 && IR2
\end{aligned}
$$

Note that nothing prevents both IR constraints from binding and then IC1 must bind, so we have $u_1 = u_2 = 0$, $u_3 = 2000$ or $w_1 = w_2 = 0$, $w_3 = 4{,}000{,}000$ as the optimal compensation to induce high effort.

Nearly identical logic applies for medium effort:

$$
\begin{aligned}
\frac{1}{4}u_1 + \frac{1}{2}u_2 + \frac{1}{4}u_3 - 200 &\geq \frac{1}{2}u_1 + \frac{1}{4}u_2 + \frac{1}{4}u_3 - 0 \\
&\equiv u_2 \geq 800 + u_1 && IC1(M \succeq L) \\
\frac{1}{4}u_1 + \frac{1}{2}u_2 + \frac{1}{4}u_3 - 200 &\geq \frac{1}{4}u_1 + \frac{1}{4}u_2 + \frac{1}{2}u_3 - 500 \\
&\equiv u_2 \geq u_3 - 1200 && IC2(M \succeq H) \\
\frac{1}{4}u_1 + \frac{1}{2}u_2 + \frac{1}{4}u_3 - 200 &\geq 0 && IR0 \\
u_1 &\geq 0 && IR1 \\
u_2 &\geq 0 && IR2 \\
u_3 &\geq 0 && IR3
\end{aligned}
$$

$IR0$ and $IR2$ are implied by other constraints, $u_1 = u_3 = 0$ satisfies the constraints, and $u_2 = 800$. Thus, to induce medium effort, wages are $w_1 = w_3 = 0, w_2 = 640{,}000$.

To induce low effort, $w_1 = w_2 = w_3 = c(L)^2 = 0$.

Remark 2b.1 For medium effort level, note that IC2 does not need to bind here since $u_2 = u_3$ minimizes risk *and* satisfies the constraint. In fact, assuming that IC2 binds leads to higher total compensation for medium effort than for high effort which isn't sensible.

Remark 2b.2 Medium effort is implementable since the two IC constraints do not contradict. A sufficiently large w_2 satisfies both constraints.

(c) If effort cannot be observed, what is the optimal contract?

This involves plugging the obtained wages into the profit equation: $e = L$:
$\frac{1}{2}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{4}\pi_3 - c(L)^2 = 3,500,000 - 0 = 3,500,000$
$e = M$: $\frac{1}{4}\pi_1 + \frac{1}{2}\pi_2 + \frac{1}{4}\pi_3 - \frac{1}{2}640,000 = 4,250,000 - 320,000 = 3,930,000$
$e = H$: $\frac{1}{4}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{2}\pi_3 - c(H)^2 = 5,250,000 - 0 = 250,000 = 5,000,000$

Checking profit for each effort level at the optimal wages:
$e = L$: $\frac{1}{2}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{4}\pi_3 - 0 = 3,500,000 - 0 = 3,500,000$
$e = M$: $\frac{1}{4}\pi_1 + \frac{1}{2}\pi_2 + \frac{1}{4}\pi_3 - \frac{1}{2}640,000 = 4,250,000 - 320,000 = 3,930,000$
$e = H$: $\frac{1}{4}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{2}\pi_3 - \frac{1}{2}4,000,000 = 5,250,000 - 2,000,000 = 3,250,000$

So inducing medium effort is best and the optimal compensation is $w_1 = w_3 = 0, w_2 = 640,000$.

Question 3. Consider a differentiated-products version of a Bertrand duopoly (firms i and j). Firm $i \neq j$ has demand given by

$$q_i = 168 - 2p_i + p_j$$

with no costs of production. Each firm's profit is $p_i q_i$.

Determine firm i's subgame-perfect equilibrium profit if:

1. The firms choose p_i and p_j simultaneously.

   Firm $i's$ profit is given by $(168 - 2p_i + p_j)p_i$. Maximizing yields the best reply: $p_i(p_j) = 42 + \frac{1}{4}p_j$ and similarly for $p_j(p_i)$. Solving, we get an equilibrium of $p_i^\star = p_j^\star = 56$. Substituting into the profit equation yields $(168 - 56)56 = 2 \times 56^2 = 6272$.

2. The firms choose p_i and p_j sequentially, with firm i choosing first.

   From above, firm $j's$ strategy (in the second period) is given by $p_j(p_i) = 42 + \frac{1}{4}p_i$. In the first period, firm i maximizes

   $$q_i p_i = (168 - 2p_i + p_j(p_i))\, p_i = \left(168 - 2p_i + (42 + \frac{1}{4}p_i)\right) p_i$$

   which yields $p_i = 60$ and, on the equilibrium path, $p_j = 57$. Substituting into the profit equation yields $(168 - 120 + 57)60 = 105 \times 60 = 6300$.

3. The firms choose p_i and p_j sequentially, with firm j choosing first.

   This yields the reverse of the above with prices on the equilibrium path given by $p_i = 57, p_j = 60$ and profit given by $(168 - 114 + 60)57 = 114 \times 57 = 6498$.