Microeconomic Theory II — Spring 2015

Midterm Exam SOLUTIONS — Mikhael Shor

GRADING Each question was graded out of 100. My handwriting is horrible, which is partly why I supply these solutions. Obviously, if you can't make out something I wrote on your exam, come see me. A solid A grade would require an average score in the mid/high 70s. Somewhere in the low 50s would separate a B from a B-. YOUR GRADE IS ON THE LAST PAGE OF YOUR BLUEBOOK/EXAM

MIDTERM SOLUTIONS Done quickly, only illustrative; they are neither complete nor guaranteed.

Question 1. Consider the following normal-form game:

		Player 2		
		R	S	T
	A	$5,5$	$8,3$	$1,1$
Player 1	B	$3,2$	$6,6$	$2,5$
	C	$2,0$	$3,0$	X,Y

(a) (25 pts) For what values of X and Y is the game above dominance solvable? This requires that (i) the game have a (strictly) dominated strategy and (ii) once that strategy is removed, the process of iterated deletion continues until a single, unique outcome remains. Note that A, R, and S cannot be dominated (they are the unique best response to R, A, and B, respectively). This leaves B, C, and T as the only possibilities.

- C is dominated (by B) if $X < 2$. Then, T, B, and S are deleted, leaving only A and R.
- T is dominated (by S or by a mixture of R and S) if $Y < 0$. Then, B, C, and S are deleted.
- It is also possible for B to be dominated by a mixture of A and C when X is sufficiently large. However, once we delete B, the iterated process would require either $X < 1$ or $Y < 0$, which are both implied by the above.

Therefore, the game is dominance solvable if $X < 2$ OR $Y < 0$. Some students were not clear that this requires one OR the other, not both conditions. This is because once either C or T is eliminated, the other is strictly dominated for any values of X or Y.

For the remainder of this question, imagine that the stage game above is repeated twice, with the outcome of the first period observed by both players before the second period, and with payoffs equal to the sum of the two periods (i.e., no discounting).

(b) (10 pts) For what values of X and Y does the game have a pure-strategy subgame-perfect Nash equilibrium in which players play $\{A,R\}$ in the first period? Since $\{A,R\}$ is an equilibrium of the stage game (for any X and Y), there *always* exists an SPNE that involves repeating the NE in every period.

(c) (50 pts) For what values of X and Y does the game have a pure-strategy subgame-perfect Nash equilibrium in which players play {B,S} in the first period?
First, $\{C,T\}$ needs to be a Nash equilibrium of the stage game. Otherwise, the stage game would have a unique NE and the repeated game would have a unique SPNE. Therefore, we require that $Y \geq 0, X \geq 2$. Second, we must have one equilibrium follow $\{B,S\}$ and another to follow $\{A,S\}$, which is the only outcome that keeps $\{B,S\}$ from being a NE. Note that other outcomes can simply be followed by the same NE as $\{B,S\}$ since they do not present an incentive conflict.

- First, consider $\{A,R\}$ following $\{B,S\}$ and $\{C,T\}$ following $\{A,S\}$. Here, $\{A,R\}$ is the "reward" for playing $\{B,S\}$ and $\{C,T\}$ is the "punishment" for playing $\{A,S\}$. Then, for B to be a best response to S in the first round, we must have $6+5 \geq 8+X$ or $3 \geq X$.
- Next, consider $\{C,T\}$ following $\{B,S\}$ and $\{A,R\}$ following $\{A,S\}$. Then, for B to be a best response to S in the first round, we must have $6+X \geq 8+5$ or $X \geq 7$.

Therefore, we must have $Y \geq 0$ and either $3 \geq X \geq 2$ or $X \geq 7$.

(d) (15 pts) For what values of X and Y does the game have a pure-strategy subgame-perfect Nash equilibrium in which players play $\{C,T\}$ in the first period? Note that if $\{C,T\}$ is not a Nash equilibrium of the stage game, then the stage game has a unique Nash equilibrium and thus a unique SPNE. If $\{C,T\}$ is a Nash equilibrium, then there exists an SPNE in which it is played in every period, including the first. Thus, there exists an SPNE in which $\{C,T\}$ is played in the first period if and only if it is a Nash equilibrium, or if $X \geq 2$ and $Y \geq 0$.

Question 2. A diamond seller can sell either a real or a fake diamond. A buyer can pay for an independent inspection of the diamond or simply buy it without inspecting it. If a seller is caught (through inspection) selling a fraudulent diamond, he must pay a fine of $F > 0$ to the government. The payoffs are:

		Buyer: Inspect	Buyer: Don't Inspect
Seller	Real	$2, 2$	$3, 3$
	Fake	$3 - F, 1$	$4, 0$

(a) (40 pts) What are the Nash equilibria of the above game (this will depend on F)?

- When $F < 1$, Fake is a dominant strategy for the seller. The unique equilibrium is {Fake, Inspect}
- When $F > 1$, the game does not have an equilibrium in pure strategies. The unique mixed-strategy equilibrium is given by $\{\frac{1}{2}\text{Fake}+\frac{1}{2}\text{Real}, \frac{1}{F}\text{Inspect}, \frac{F-1}{F}\text{Don't Inspect}\}$
- When $F = 1$, the above are both equilibria. In fact, because the best response to Inspect is not unique, there exist a multiplicity of equilibria of the form $\{\ p\ \text{Real}+(1-p)\ \text{Fake, Inspect}\ \}$ for $p \leq \frac{1}{2}$. When the Buyer inspects, the seller is indifferent between Real and Fake. When the seller selects Real less than half of the time, the Inspect is a best response for the buyer.

(b) (40 pts, including (c) below) What are the seller's and the buyer's expected equilibrium payoffs?

- When $F \leq 1$, payoffs are $\{3 - F, 1\}$.
- When $F \geq 1$, the seller's payoff is $2\frac{1}{F} + 3\frac{F-1}{F} = 3 - \frac{1}{F}$. Note that this makes intuitive sense; since the seller earns between 2 and 3 when playing Real, and the seller must be indifferent between his two strategies, the expected equilibrium payoff is always between 2 and 3. The buyer's expected payoff is $2\frac{1}{2} + 1\frac{1}{2} = \frac{3}{2}$.
- When $F = 1$, the Seller earns 2 in every equilibrium, and the buyer earns between 1 and $\frac{3}{2}$, depending on the equilibrium.

(c) How do the buyer's and seller's equilibrium payoffs change with F?

- The seller's profit initially decreases from 3 to 2 (as F goes from 0 to 1) and then increases asymptotically towards 3.
- The buyer's payoff jumps from 1 (for $F < 1$) to $\frac{3}{2}$ (for $F > 1$).

(d) (20 pts) The government's expected revenue is equal to the fine, F, times the equilibrium probability that the fine is collected (when the buyer inspects a fake diamond). Describe or sketch the government's expected

revenue as a function of F. When $F < 1$, the probability of collecting the fine is 1, and therefore revenue is F. For $F > 1$, expected revenue is $\Pr\{Fake\}\Pr\{Inspect\}F = \frac{1}{2}\frac{1}{F}F = \frac{1}{2}$. When $F = 1$, again every revenue between $\frac{1}{2}$ and 1 is possible for some equilibrium. $F > 1$).
For some reason, a number of students answered (c) and (d) with reference only to either the pure-strategy equilibrium ($F < 1$) or the mixed-strategy equilibrium ($F > 1$) and missed the fact that the nature of the payoffs changes dramatically around $F = 1$.

Question 3. Consider n identical fishermen who fish in the same lake. Every year, each fisherman receives an income of 1 and must choose how much of this income to contribute to stocking the lake with fish for the upcoming season. There is no saving and contribution decisions are made simultaneously in each year. Stocking the lake helps all fishermen; for every dollar invested in stocking fish, *each* fisherman receives α dollars in additional revenue, $\frac{1}{n} < \alpha < 1$ (e.g., $\alpha = 0.7$).

Denote by m_i, $0 \leq m_i \leq 1$, fisherman i's contribution. Thus, the utility of fisherman i is:

$$u_i = 1 - m_i + \alpha \sum_{j=1}^{n} m_j$$

(a) (30 pts) Find the Nash equilibrium of this game.
Utility is given by

$$u_i = 1 - m_i + \alpha \sum_{j=1}^{n} m_j = 1 - (1-\alpha)m_i + \alpha \sum_{j \neq i} m_j$$

and thus $\frac{du_i}{dm_i} = -(1-\alpha) < 0$ so the equilibrium is $m_i = 0$, $\forall i$.

(b) (15 pts) Find the socially-optimal level of investment for each fisherman (i.e., the level of investment by each that maximizes total welfare).

$$\sum_{i=1}^{n} u_i = n - \sum_{i=1}^{n} m_i + \alpha \sum_{i=1}^{n}\sum_{j=1}^{n} m_j = n + (\alpha n - 1)\sum_{i=1}^{n} m_i$$

which is increasing in m_i. Therefore, the socially-optimal investment is $m_i = 1$, $\forall i$.

(c) (50 pts) Suppose that this game is infinitely repeated, with future payoffs discounted by δ. Each fisherman adopts the following strategy:

- Contribute 1 in the first year and in every subsequent year if, in all previous years, everyone contributed 1.
- Contribute 0 If, in any previous year, anyone contributed less than 1.

For what range of δ are these strategies a subgame-perfect Nash equilibrium of the infinitely-repeated game?

Assume other fisherman follow the above strategy.
If fisherman i cooperates and also contriubutes 1, he earns αn in each period. If fisherman i deviates, he contributes 0 in period 1, earning $1 + \alpha(n-1) = \alpha n + (1-\alpha)$. But then all contribute 0 in each future period, resulting in payoffs of 1 in each future period. For cooperation to be a best response:

$$\frac{\alpha n}{1-\delta} \geq \alpha n + (1-\alpha) + \frac{\delta}{1-\delta}$$

or

$$\delta \geq \frac{1-\alpha}{\alpha(n-1)}$$

which is less than 1 (since $\alpha > \frac{1}{n}$).

(d) (5 pts) How does your answer in (c) change with n? Does this make sense? Why or why not?

The critical discount rate is decreasing in n which means that cooperation is getting *easier* with more fishermen. This runs contrary to our usual intuition that cooperation is harder with more parties as there is more free-riding. Here, the reason for this is actually because of a silly modeling assumption. Note that, for a contribution of 1, a fisherman provides α of value for *each* other fisherman. This means that a contribution of 1 leads to a total of αn in value, which is growing (and unbounded) with n. This seems strange; fish are a rivalrous good, so if I stock a river with fish, the total amount I contribute should be (more or less) fixed. This is why many models of public goods replace α in the above with $\frac{\alpha}{n}, \alpha > 1$. In this formulation, α is interpreted as my total contribution to welfare (independent of n).