Microeconomic Theory II **Spring 2021**

Midterm Exam **Mikhael Shor**

Solutions

Question 1. Consider the following normal-form game:

		Player 2		
		X	Y	Z
Player 1	A	α, β	$10, 10$	$30, 10$
	B	$5, 3$	$25, 20$	$40, 10$
	C	$8, 8$	$15, 16$	$25, 18$

(a) For what values of α is the strategy A strictly dominated?

ANSWER: $\alpha < 7$.

A is dominated by B but not by C. We check if A is dominated by a mixture of B and C, $pB + (1-p)C$. This would require:

- $X : 5p + 8(1-p) > \alpha$
- $Y : 25p + 15(1-p) > 10$ which is always true.
- $Z : 40p + 25(1-p) > 30$ which is true when $p > \frac{1}{3}$

Thus, A is dominated when there exists a $p > \frac{1}{3}$ for which $\alpha < 5p + 8(1-p) \leq 5(\frac{1}{3}) + 8(\frac{2}{3}) = 7$.

NOTE 1.a.1 : The critical part of this question is remembering that a pure strategy can be dominated by a mixed strategy. Additionally, some students, whether recognizing that a mixed strategy can dominate A here or not, made some erroneous comments about causality (implying, for example, that we should look for a mixed strategy *because* we found a pure strategy that dominates A.) A strategy can be dominated by a mixed strategy even if it is not dominated by a pure strategy.

For the rest of the question, imagine that the stage game above is repeated twice, with the outcome of the first period observed by both players before the second period, and with payoffs equal to the sum of the two periods (i.e., no discounting).

(b) For what values of α and β does the game have a pure-strategy subgame-perfect Nash equilibrium in which players play {B,Y} in the first period? Explain.

ANSWER: for all α and β.

$\{B, Y\}$ is a Nash equilibrium of the stage game for any α and β. There always exists an SPNE in which a stage-game NE is repeated at every stage.

NOTE 1.b.1 : Note that $\{B, Y\}$ being a NE of the stage game is sufficient here. Whether or not it is unique is not relevant to whether there is an SPNE that involves playing it in the first period (that is relevant only to the question of whether that SPNE is also unique).

(Note: for the remaining questions, since the question asks only about pure-strategy Nash equilibria, all discussions of number of equilibria refer to pure-strategy equilibria)

(c) For what values of α and β does the game have a pure-strategy subgame-perfect Nash equilibrium in which players play {A,X} in the first period? Explain.

ANSWER: $\alpha \geq 8, \beta \geq 10$.

For $\{A, X\}$ to be played in the first period, one of two things has to be true: either $\{A, X\}$ is a Nash equilibrium of the stage game or there exist at least two other Nash equilibria that can be used conditionally in the second period to make $\{A, X\}$ mutual best responses in the first period. Here, if $\{A, X\}$ isn't a NE of the stage game, the stage game has a unique equilibrium. Thus, the only way is for $\{A, X\}$ to be a NE of the stage game.

NOTE 1.c.1 : For completeness, students who noted that $\{A, X\}$ needs to be a NE of the stage game should clarify that this is not a general statement but because otherwise the stage game would have a unique NE.

(d) For what values of α and β does the game have a pure-strategy subgame-perfect Nash equilibrium in which players play {A,X} in the *second* period? Explain.

ANSWER: $\alpha \geq 8, \beta \geq 10$.

The second period is the final period and thus the subgame starting in the second period is the stage game. Only a NE of the stage game can be part of a SPNE in the final period.

NOTE 1.d.1 : Note here (and elsewhere) that the inequalities are weak, not strict. Best replies (and thus equilibria) do not require strict inequalities.

(e) For what values of α and β does the game have a pure-strategy subgame-perfect Nash equilibrium in which players play {C,Z} in the first period? Explain.

ANSWER: $\beta \geq 10$ and either $8 \leq \alpha \leq 10$ or $\alpha \geq 40$.

Since $\{C, Z\}$ is not a NE of the stage-game (specifically, because Player 1 would prefer both A or B in response to Z), we require one stage-game equilibrium to follow $\{C, Z\}$ in the first period and another to follow $\{A, Z\}$ and $\{B, Z\}$ such that, in the first period, C becomes a best response to Z. Note that Player 2's incentives are easier since Z is already a best reply to C. Specifically, if $\{C, X\}$, $\{C, Y\}$, and $\{C, Z\}$ in the first period are all followed by the same stage-game NE in the second period, then Player 2's best response remains Z.

First, we require that $\{A.X\}$ be a Nash equilibrium of the stage game (otherwise, the stage game has a unique NE and only it can be part of a SPNE). Thus, $\alpha \geq 8$ and $\beta \geq 10$.

Second, we can either use $\{A.X\}$ as the "reward" for playing $\{C, Z\}$ and $\{B, Y\}$ as the "punishment" or vice versa.:

- $\{A.X\}$ as the "reward": We need $\{C, Z\}$ followed by $\{A, X\}$ to be preferred by player 1 to $\{B, Z\}$ followed by $\{B, Y\}$ (which also implies that it is preferred to $\{A, Z\}$ followed by $\{B, Y\}$):
$25 + \alpha \geq 40 + 25$ or $\alpha \geq 40$.
- $\{A.X\}$ as the "punishment": We need $\{C, Z\}$ followed by $\{B, Y\}$ to be preferred by player 1 to $\{B, Z\}$ followed by $\{A, X\}$ (and to $\{A, Z\}$ followed by $\{A, X\}$):
$25 + 25 \geq 40 + \alpha$ or $\alpha \leq 10$.

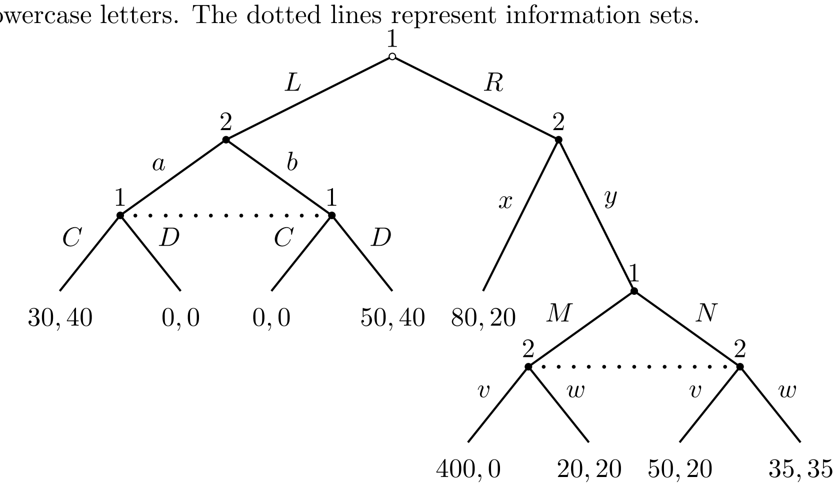
Combining the above conditions yields the answer.

NOTE 1.e.1 : Note that, aside from ensuring that $\{A, X\}$ is an equilibrium, no additional conditions on β are necessary. This is because player 2 does not require a reward and punishment to play Z in response to C. Since Z is a best response in the stage game, it remains a best response if every payoff for player 2 in the bottom row is incremented by the same amount.

NOTE 1.e.2 : A number of students considered $\{A, X\}$ as either the reward or the punishment but not both. What is required in this question is for player 1's payoffs from $\{A, X\}$ and from $\{B, Y\}$ to be sufficiently far apart (at least 15) to offset the incentive to deviate to B in response to Z. That can be done by making α sufficiently small ($\leq 25 - 15$) or sufficiently big ($\geq 25 + 15$).

NOTE 1.e.3 : The payoffs from the other stage game NE, $\{B, Y\}$ are irrelevant to this question. Whether something is an equilibrium depends only on whether each player is playing a best response and not on whether the players earn more than they do at some other equilibrium.

Question 2. Consider the game below. Player 1's actions are denoted by uppercase letters and Player 2's actions by lowercase letters. The dotted lines represent information sets.



(a) Describe the strategy space (or provide an example of a pure strategy) for each player.

ANSWER: Each player has three information sets with two actions at each information set. The strategy space for player 1 is $\{L, R\} \times \{C, D\} \times \{M, N\}$ (e.g., L, C, M) and the strategy space for player 2 is $\{a, b\} \times \{x, y\} \times \{v, w\}$ (e.g., a, x, v).

(b) Find *all* subgame-perfect Nash equilibria. For each equilibrium, also write down each player's equilibrium payoffs.

ANSWER: $\{R, C, N; a, y, w\}, \{L, D, N; b, y, w\}, \{R, \frac{1}{2}C + \frac{1}{2}D, N; \frac{5}{8}a + \frac{3}{8}b, y, w\}$ with corresponding payoffs of $\{35, 35\}, \{50, 40\}$, and $\{35, 35\}$.

We proceed by backwards induction. Consider the subgame on the left:

Player 1 \ Player 2	a	b
C	30, 40	0, 0
D	0, 0	50, 40

There are three Nash equilibria of this subgame: $\{C, a\}, \{D, b\}$, and $\{\frac{1}{2}C + \frac{1}{2}D, \frac{5}{8}a + \frac{3}{8}b\}$.

The smaller subgame on the right has a unique Nash equilibrium: $\{N, w\}$ and player 2 choosing between x (20) and Y (followed by N and w for 35) prefers y.

By backwards induction, player 1 then selects L or R by comparing the payoffs of each of the three NE of the left subgame with the payoff from R which is 35.

NOTE 2.b.1 : Note that one could proceed by finding the NE of each subgame, including the game as a whole, and then seeing which Nash Equilibria induce equilibria in each subgame, but since the question didn't ask about NE, this is a much lengthier process.

NOTE 2.b.2 : There are three PSNE of the left subgame and one PSNE of the right subgame. Since player 1 is never indifferent between any of the equilibria of the left subgame and the one equilibrium of the right subgame, this yields precisely three SPNE.

NOTE 2.b.3 : In calculating the payoffs from the mixed strategy NE of the left subgame, recall that each strategy should yield the same expected payoffs. For player 1: $U_1(C) = 30\frac{5}{8} = \frac{75}{4} = 50\frac{3}{8} = U_1(D)$ so equilibrium payoffs are $\frac{75}{4}$. For player 2: $U_2(a) = \frac{1}{2}40 = 20 = \frac{1}{2}40 = U_2(b)$ so equilibrium payoffs are 20.

Question 3. Consider two firms (Firm 1 and Firm 2) who are Cournot competitors (compete in quantities). Inverse demand is given by $p = 100 - q_1 - q_2$. Each firm has costs given by $c(q_i) = 10q_i$, $i \in \{1, 2\}$ (i.e., a marginal cost of 10).

Before the two firms play the Cournot game, Firm 1 can invest in a cost-eliminating technology. If it invests, its costs become $c(q_1) = 0$. The cost of this investment is F.

The timing of the game is:

- In the first period, Firm 1 chooses to invest (at a cost of F) or to not invest. Firm 2 observes Firm 1's choice.
- In the second period, Both firms simultaneously select a quantity, q_i. Firm 2's cost is given by $c(q_2) = 10q_2$. Firm 1's cost is given by $c(q_1) = 10q_1$ if it chose not to invest and by $c(q_1) = 0$ if it chose to invest.

Profits for each firm are given by $pq_i - c(q_i)$ minus the cost of the investment for Firm 1, F, if Firm 1 chooses to invest.

(a) Carefully identify the subgame-perfect Nash equilibrium of this game (this will depend on F).

ANSWER:
If $F < \frac{4000}{9}$: $\left\{\text{Invest}, q_1^N = 30, q_1^I = \frac{110}{3}; q_2^N = 30, q_2^I = \frac{80}{3}\right\}$
If $F > \frac{4000}{9}$: $\left\{\text{No Invest}, q_1^N = 30, q_1^I = \frac{110}{3}; q_2^N = 30, q_2^I = \frac{80}{3}\right\}$
If $F = \frac{4000}{9}$: $\left\{p\text{Invest} + (1-p)\text{No Invest}, q_1^N = 30, q_1^I = \frac{110}{3}; q_2^N = 30, q_2^I = \frac{80}{3}\right\}, \forall p \in [0, 1]$
where superscripts I, N refer to actions if firm 1 chooses Invest and No Invest, respectively.

First, we identify the Nash equilibria of each of the two proper subgames (following when firm 1 doesn't invest and when firm 1 does invest).

- When player 1 does not invest, profits for the two firms are given by:

$$\pi_1(q_1, q_2) = (100 - q_1 - q_2 - 10)q_1, \quad \pi_2(q_1, q_2) = (100 - q_1 - q_2 - 10)q_2$$

  And best replies are given by

$$q_1(q_2) = 45 - \frac{1}{2}q_2, \quad q_2(q_1) = 45 - \frac{1}{2}q_1$$

  which yields a NE of $q_1 = q_2 = 30$ and NE profits of 900 for each firm.
- When player 1 does invest, profits for the two firms are given by:

$$\pi_1(q_1, q_2) = (100 - q_1 - q_2)q_1, \quad \pi_2(q_1, q_2) = (100 - q_1 - q_2 - 10)q_2$$

  And best replies are given by

$$q_1(q_2) = 50 - \frac{1}{2}q_2, \quad q_2(q_1) = 45 - \frac{1}{2}q_1$$

  which yields a NE of $q_1 = \frac{110}{3}, q_2 = \frac{80}{3}$ and NE profits for firm 1 of $\left(\frac{110}{3}\right)^2$.

Next, we identify whether firm 1 elects to invest by comparing its profits from investing and not investing. Investing is (weakly) preferred if:

$$\left(\frac{110}{3}\right)^2 - F \geq 900$$

or if $F \leq \frac{4000}{9}$.

NOTE 3.a.1 When firm 1 is indifferent between investing and not investing, any random combination between the two is a best response.

NOTE 3.a.2 Note that a strategy for a firm includes its actions (i.e., the quantity) in *each* subgame.

NOTE 3.a.3 Following the investment decision, each firm has a single information set and thus a strategy in each proper subgame is simply a number (the equilibrium quantity) and not a function conditional on the other player's strategy (that's the best reply).

(b) What is the maximum value of F for which it is subgame-perfect for Firm 1 to invest in the technology?

ANSWER: from above, $\frac{4000}{9}$

(c) ** Describe a Nash equilibrium with an F greater than that identified in (b) for which investing is not part of a subgame-perfect Nash Equilibrium but is part of a Nash equilibrium.

This question was not graded but tests for a deeper understanding of the difference between a Nash equilibrium and a subgame-perfect Nash equilibrium. A Nash equilibrium allows an action in a subgame that is unreached in equilibrium to be different than the action one would actually take there if it were reached. That is, in a Nash equilibrium, a player is indifferent between all strategies that differ only at unreached subgames.

Consider firm 2 playing the following strategy: if player 1 invests, then $q_2 = \frac{80}{3}$ and if player 1 does not invest, play $q_2 = 90$. In this case, if Firm 1 invests, it expects profits (as above) of $\left(\frac{110}{3}\right)^2 - F$ and if it doesn't invest, it expects profits of 0 since q_2 is too high to allow firm 1 to profit from any positive q_1. Thus, if $F = 500$ for example, we have $500 > \frac{4000}{9}$ which means that it would not invest in the subgame perfect equilibrium, but we have $\left(\frac{110}{3}\right)^2 - 500 > 0$ which means that it is a best reply to invest given the actions of firm 2.
Basically, Firm 2 is best-responding to Firm 1 but without any limitation on what to do in the (non-equilibrium) case of No Investment. SPNE here forces Firm 2 in equilibrium to play a best response in both the case of investing and not investing.