Microeconomic Theory II **Spring 2022**

Midterm Exam **Mikhael Shor**

Solutions

Question 1. Consider the following normal-form game:

		Player 2			
		W	X	Y	Z
Player 1	A	2, 6	8, 0	0, 1	4, 1
	B	6, 0	4, 6	6, 1	0, 0
	C	0, 10	0, 0	6, 6	0, 0
	D	2, 0	2, 10	$\boldsymbol{\alpha}$, 6	2, 10

(a) Assume $\boldsymbol{\alpha} = 5$. Which strategies are a best response for *some* beliefs (i.e., consistent with rationality)? For each, specify a belief or show that none exists.

All strategies are best reply to some beliefs.
- For player 1, A is a best response to X; B is a best response to W or Y, C is a best response to Y. D is a best reply to (for example), $\frac{1}{2}Y + \frac{1}{2}Z$.
- For player 2, W is a best response to A, X is a best response to B or D, Z is a best response to D, and Y is a best response to (for example) $\frac{1}{2}C + \frac{1}{2}D$.

NOTE 1.a.1: Some students found only strategies that are a best response to some pure strategy and assumed that all other strategies are never a best response. The critical part of this question is remembering that a pure strategy might not be a best response to any pure strategy but still be a best response to a mixed strategy. Further, a few students simply listed best replies to pure strategies and made no mention of strategies D and Y.

NOTE 1.a.2: Some students noted that D or Y are not strictly dominated or are a best reply to some beliefs, but did not provide an example of what they are best replies to.

NOTE 1.a.3: This question addresses only the assumption of rationality and thus there is no need to reference or somehow attempt to apply the assumption of common knowledge.

(b) For what values of $\boldsymbol{\alpha}$ is D strictly dominated?

For D to be strictly dominated, some mix of A, B, and C must earn strictly higher payoffs for any pure strategy of player 2. Further, since B weakly dominates C, we can consider whether D is dominated by a mix of just A and B:
- W: $2\sigma_A + 6\sigma_B > 2 \Rightarrow \sigma_A < 1$
- X: $8\sigma_A + 4\sigma_B > 2$ [always true]
- Y: $6\sigma_B > \alpha \Rightarrow 6 - 6\sigma_A > \alpha$
- Z: $4\sigma_A > 2 \Rightarrow \sigma_A > \frac{1}{2}$

Therefore, D is dominated whenever there exists a $\sigma_A, \frac{1}{2} < \sigma_A < 1$, such that $\alpha < 6 - 6\sigma_A$,

or $\alpha < 3$.
We can confirm this by:
- first, considering the strategy $(\frac{1}{2}+\epsilon)A + (\frac{1}{2}-\epsilon)B$ which dominates D when $\alpha < 3$
- second, confirming that no strategy can dominate D when $\alpha = 3$ since that would require probability strictly greater than $\frac{1}{2}$ on both A and B.

NOTE 1.b.1: $\frac{1}{2}A + \frac{1}{2}B$ does not dominate D since, in the case of player 2 playing Z, both strategies earn 2.

NOTE 1.b.2: Some students claimed that D can never be dominated by a strategy that puts positive probability on C. This is not correct. However, for any strategy that puts positive probability on C, there is a strategy that earns at least as much by shifting that probability to B.

NOTE 1.b.3: A student should identify all values of α for which D is dominated. Some students checked when D is dominated by just one arbitrary mixed strategy.

NOTE 1.b.4: Some students, instead of noting that D is dominated whenever there exists a σ_A, $\frac{1}{2} < \sigma_A < 1$, such that $\alpha < 6 - 6\sigma_A$, or $\alpha < 3$, assumed that D is dominated only when $\alpha < 6 - 6\sigma_A$ is true for all σ_A, $\frac{1}{2} < \sigma_A < 1$, yielding $\alpha < 0$. This is, intuitively, a strange thing to calculate as it equates to making sure that whenever D earns less than some other strategy when player 2 plays W, X, or Z, then D earns less than that other strategy when player 2 plays Y.

(c) Assume $\boldsymbol{\alpha}$ is a value for which D is strictly dominated. Which strategies survive the iterated deletion of strictly dominated strategies? Explain each step.

- D is dominated by assumption (or by $(\frac{1}{2}+\epsilon)A + (\frac{1}{2}-\epsilon)B$)
- Y and Z are dominated by, e.g., $\frac{3}{4}W + \frac{1}{4}X$
- C is dominated by both A and B

No further strategies can be eliminated. Therefore, strategies A,B,W,X remain.
Several orders of elimination are possible, including D→Z→Y→C and D→Y→C→Z.

NOTE 1.c.1: While several orders of elimination are possible, some are not. For example, Y is not strictly dominated until after D is eliminated.

Question 2. Consider the two-firm Stackelberg model but with firms sequentially setting prices instead of quantities. First, firm 1 sets a price p_1. Next, after observing p_1, firm 2 sets a price, p_2. As in the Bertrand model, the firm with the lower price sells the entire market demand, $Q = Q(\min\{p_1, p_2\})$, while the other firm sells zero. If both prices are equal, market demand is split evenly between the two firms, $q_1 = q_2 = Q/2$. Assume constant marginal costs of c and assume $Q(c) > 0$.

(a) What are the (pure strategy) subgame-perfect Nash equilibria of the above game?

`Consider firm 2's decision following firm 1. Generally, firm 1 will want to undercut firm 2 by some arbitrarily small amount. However, a more careful consideration of firm 2's best responses yields:`

$$p_2(p_1) = \begin{cases} \text{any } p > p_1, & \text{if } p_1 < c & \text{(at prices below cost, make sure } q_2 = 0) \\ \text{any } p \geq c, & \text{if } p_1 = c & \text{(indifferent between } p = c \text{ and } q_2 = 0) \\ p_1 - \epsilon, & \text{if } c < p_1 \leq p^M & \text{(undercut slightly so that } q_2 = Q) \\ p^M, & \text{if } p^M > p_1 & \text{(don't exceed some optimal `monopoly' price)} \end{cases}$$

`Because each` p_1 `defines a subgame, a strategy for firm 2 specifies a` p_2 `for every possible` p_1`.`

For firm 1, it must consider the profit from every possible p_1 given the reply of firm 2:

if firm 1 sets:	firm 2 responds:	firm 1 earns:
$p_1 < c$	$p_2 > p_1$	$(p_1 - c)Q(p_1) < 0$
$p_1 = c$	$p_2 \geq c$	$(c - c)q_1 = 0$
$c < p_1$	$p_2 < p_1$	$(p_1 - c)0 = 0$

Therefore, firm 1 can never earn positive profits. It is indifferent between any $p_1 \geq c$. Any $p_1 \geq c$ and any $p_2(p_1)$ consistent with the best responses above is a subgame-perfect Nash equilibrium. For concreteness, the following is one example of a SPNE:

$$p_1 = 2c, p_2(p_1) = \begin{cases} p_1 + 1, & \text{if } p_1 \leq c \\ p_1 - \epsilon, & \text{if } c < p_1 \leq p^M \\ p^M, & \text{if } p^M > p_1 \end{cases}$$

Firm 2 undercuts when it is profitable to do so; firm 1 selects a price (among many) that yields it zero profit.

NOTE 2.a.1: It is very important to consider best replies carefully and completely and not just "intuitively." This would reveal that (i) $p_2 < c$ can be a best reply whenever $p_2 > p_1$, that (ii) best replies to $p_1 \leq c$ are not unique (as firm 2 is indifferent between any price that results in zero profit), and (iii) there is a maximum p_2 consistent with profit maximization. All of this is identical to our discussion of best replies in the standard (simultaneous) Bertrand model.

NOTE 2.a.2: A strategy must specify an action at every information set (in every subgame for firm 2). For example, $p_2 = c$ may be a best response to $p_1 = c$, but it is not a complete strategy for firm 2. An SPNE consists of a full description of player 2's strategy. Similarly, $p_1 = c, p_2 = c$ fails to capture the multiplicity of player 2's best responses. in the simultaneous game, $p_1 = c, p_2 > c$ cannot be an equilibrium since firm 1's best response to p_2 would not be c. In the sequential game, however, these can be supported by a SPNE!

NOTE 2.a.3: After calculating firm 2's best responses to firm 1, some students then considered what firm 1's best response to firm 2 would be, suggesting, for example, that firm 1 would want to set a price that undercuts firm 2's price. This is the logic of how we would determine a Nash equilibrium if prices were set simultaneously, but is not the way to determine the subgame-perfect Nash equilibrium when prices are set sequentially. Specifically, firm 1's prices do not need to be best replies to firm 2's prices but instead need to be optimal *given* how firm 2 will respond. This is they key distinction between the traditional Bertrand model and this question. A few students didn't explicitly reference firm 1's best responses but just asserted that firm 1 will set a $p_1 = c$ if it anticipates firm 2's responses.

NOTE 2.a.4: Some students determined firm 2's best responses for each subgame (each p_1) and then called each of those best responses an SPNE. This fails to maximize firm 1's profit *given* firm 2's best responses.

(b) Briefly describe and explain if this differs from the outcome of the simultaneous Bertrand game (in which firms set prices simultaneously).

While the outcome in which both firms set a price equal to cost is an equilibrium in both the simultaneous and sequential version, it is just one of many outcomes in the sequential version. In the Bertrand model, best responding to the other firm's price leaves the other firm with zero or negative profit. In the simultaneous case, since both must be playing a best response, both earn zero profit. In the sequential case, firm 1's price does not need to be a best reply to firm 2's price and thus firm 1 is indifferent between any price that firm 2 will ultimately undercut (or already earns zero profit).

Question 3. Two coworkers are competing for a promotion to be awarded on Wednesday (in two days). The boss wants to award the promotion to whichever worker works hardest the next two days. Each coworker must decide how many of the next two days they want to stay late at work and work through the night. Each worker decides to stay late 0 days, 1 day, or 2 days ($S_i = \{0, 1, 2\}$). Utility decreases by 1 for every day a worker stays late. Assume that the two workers decide on the number of days to stay late simultaneously.

If both workers stay late an equal number of days, the promotion is awarded to one of them at random, with equal probability. Otherwise, the worker who stayed late more days gets the promotion. The promotion provides a utility of 3 while no promotion provides a utility of 0.

For example, if one worker stays late 1 day and the other stays late 2 days, the worker who stayed late 1 day receives utility of $0 - 1 = -1$ while the worker who stayed late 2 days receives utility of $3 - 2 = 1$.

(a) Carefully write down the normal form (strategic form) of this game.

`payoffs are given by:`

$$u_i = \begin{cases} 3 - s_i, & \text{if } s_i > s_j \\ 0 - s_i, & \text{if } s_i < s_j \\ \frac{1}{2}3 + \frac{1}{2}0 - s_i, & \text{if } s_i = s_j \end{cases}$$

`and the normal form by:`

		Player 2: 0	Player 2: 1	Player 2: 2
Player 1	0	$\frac{3}{2}, \frac{3}{2}$	$0, 2$	$0, 1$
	1	$2, 0$	$\frac{1}{2}, \frac{1}{2}$	$-1, 1$
	2	$1, 0$	$1, -1$	$-\frac{1}{2}, -\frac{1}{2}$

NOTE 3.a.1: Some people miscalculated the payoffs for reasons that weren't always easy to identify. Note, as a test, that the total payoffs in every outcome should equal $3 - s_i - s_j$ since the promotion is worth 3 to whichever player receives it and each day staying late has a disutility of 1.

(b) Which strategies are consistent with rationality? Briefly explain. `All. 1 is a best response to 0, 2 is a best response to 1, and 0 is a best response to 2.`

(c) Which strategies survive the iterated deletion of dominated strategies? Briefly explain. `All. None is strictly dominated.`

(d) Show that each player using each strategy with equal probability is a Nash equilibrium. `If one player assigns probability` $\frac{1}{3}$ `to each pure strategy, the other player's expected payoffs are:`
`- from 0:` $\frac{1}{3}(\frac{3}{2}) + \frac{1}{3}(0) + \frac{1}{3}(0) = \frac{1}{2}$
`- from 1:` $\frac{1}{3}(2) + \frac{1}{3}(\frac{1}{2}) + \frac{1}{3}(-1) = \frac{1}{2}$
`- from 2:` $\frac{1}{3}(1) + \frac{1}{3}(1) + \frac{1}{3}(-\frac{1}{2}) = \frac{1}{2}$
`Thus, any mixed strategy (including the one assigning equal probability to each pure strategy) is a best reply.`

NOTE 3.d.1: The question asked you merely to confirm that this is an equilibrium, requiring only a showing that both players are playing a best response (are indifferent). It did not ask to derive the equilibrium or to check if the equilibrium is unique.

(e) Now imagine that the game is played sequentially. On the first day, each player decides whether to stay late or not. On the second day (after observing the outcome of day 1), each again decides whether to stay late or not. Like before, the worker who stays late more days receives the promotion (utility of 3) or, in case of a tie, the promotion is awarded randomly (utility to each of $1\frac{1}{2}$). What is/are the subgame perfect Nash equilibrium/equilibria?

NOTE: This question was graded as optional due to some students facing time constraints.

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On the second day, there are four possible subgames.
- Subgame 1:  day 2 if neither worker worked late on day 1:
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		Player 2: don't stay late	Player 2: stay late
Player 1	don't stay late	$\frac{3}{2},\frac{3}{2}$	$0,2$
	stay late	$2,0$	$\frac{1}{2},\frac{1}{2}$

The equilibrium is { stay late ; stay late } with payoffs of $\frac{1}{2}$ for each player.
- Subgame 2: day 2 if both workers worked late on day 1:

		Player 2: don't stay late	Player 2: stay late
Player 1	don't stay late	$\frac{1}{2},\frac{1}{2}$	$-1,1$
	stay late	$1,-1$	$-\frac{1}{2},-\frac{1}{2}$

The equilibrium is { stay late ; stay late } with payoffs of $-\frac{1}{2}$ for each player.
Subgame 3 (and 4 by symmetry): day 2 if worker 1 stayed late and worker 2 did not stay late on day 1:

		Player 2: don't stay late	Player 2: stay late
Player 1	don't stay late	$2,0$	$\frac{1}{2},\frac{1}{2}$
	stay late	$1,0$	$1,-1$

The equilibrium is { $\frac{1}{3}$late ; $\frac{2}{3}$late } with payoffs of 1 and 0.
Given the above, the first day looks like this:

		Player 2: don't stay late	Player 2: stay late
Player 1	don't stay late	$\frac{1}{2},\frac{1}{2}$	$0,1$
	stay late	$1,0$	$-\frac{1}{2},-\frac{1}{2}$

which has three NE, and thus the whole game has three SPNE.

NOTE 3.e.1: A number of students stated that since the original game had a unique equilibrium identified in part d, then this game has a unique SPNE. Note that this is true for finite repetitions of a stage game. However, this is not just a repetition of the original game, but an entirely different game, where the payoffs in the second stage depend on the actions taken in the first stage.