Appendix to Shor, Kurtulus, and Galbreth (2015)

1 Conditional expectations and variances

The variance-covariance matrix for $(\mathrm{D}, \Psi_R, \Psi_S)$ is given by:

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma^2 & \sigma^2 & \sigma^2 \\ \sigma^2 & \sigma^2+\sigma_R^2 & \sigma^2+\rho\sigma_R\sigma_S \\ \sigma^2 & \sigma^2+\rho\sigma_R\sigma_S & \sigma^2+\sigma_S^2 \end{pmatrix} = \left(\sigma^2\right)_{3\times 3} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_R^2 & \rho\sigma_R\sigma_S \\ 0 & \rho\sigma_R\sigma_S & \sigma_S^2 \end{pmatrix}$$

Throughout, we make use of the properties of the conditional distributions $\mathrm{D}|\Psi_R$ and $\mathrm{D}|\Psi_R, \Psi_S$. The first two are used in the non-collaborative model, and the last in the collaborative models.

$$\mathbb{E}[\mathrm{D} \mid \Psi_R = \psi_R] = \mathbb{E}[\mathrm{D}] + \frac{Cov[\mathrm{D}, \Psi_R]}{\mathbb{V}[\Psi_R]}(\psi_R - \mathbb{E}[\Psi_R]) \tag{1}$$

$$= \mu\left(\frac{\sigma_R^2}{\sigma^2+\sigma_R^2}\right) + \psi_R\left(\frac{\sigma^2}{\sigma^2+\sigma_R^2}\right) \tag{2}$$

$$\mathbb{V}[\mathrm{D} \mid \Psi_R = \psi_R] = \mathbb{V}[\mathrm{D}] - \frac{Cov[\mathrm{D}, \psi_R]^2}{\mathbb{V}[\Psi_R]} \tag{3}$$

$$= \sigma^2\left(\frac{\sigma_R^2}{\sigma^2+\sigma_R^2}\right) \tag{4}$$

$$\mathbb{E}[\mathrm{D} \mid \Psi_R = \psi_R, \Psi_S = \psi_S] = \mathbb{E}[\mathrm{D}] + \begin{pmatrix} \mathbf{\Sigma}_{1,2} & \mathbf{\Sigma}_{1,3} \end{pmatrix} \begin{pmatrix} \mathbf{\Sigma}_{2,2} & \mathbf{\Sigma}_{2,3} \\ \mathbf{\Sigma}_{3,2} & \mathbf{\Sigma}_{3,3} \end{pmatrix}^{-1} \begin{pmatrix} \psi_R - \mathbb{E}[\Psi_R] \\ \psi_S - \mathbb{E}[\Psi_S] \end{pmatrix} \tag{5}$$

$$= \frac{w_R\psi_R + w_S\psi_S + w_\mu\mu}{w_R + w_S + w_\mu} \tag{6}$$

$$\mathbb{V}[\mathrm{D} \mid \Psi_R = \psi_R, \Psi_S = \psi_S] = \mathbb{V}[\mathrm{D}] - \begin{pmatrix} \mathbf{\Sigma}_{1,2} & \mathbf{\Sigma}_{1,3} \end{pmatrix} \begin{pmatrix} \mathbf{\Sigma}_{2,2} & \mathbf{\Sigma}_{2,3} \\ \mathbf{\Sigma}_{3,2} & \mathbf{\Sigma}_{3,3} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{\Sigma}_{2,1} \\ \mathbf{\Sigma}_{3,1} \end{pmatrix} \tag{7}$$

$$= \frac{w_\mu\sigma^2}{w_R + w_S + w_\mu} \tag{8}$$

where $w_R = \sigma^2(\sigma_S^2 - \rho\sigma_R\sigma_S)$, $w_S = \sigma^2(\sigma_R^2 - \rho\sigma_R\sigma_S)$, and $w_\mu = (1-\rho^2)\sigma_R^2\sigma_S^2$.

As we focus on the case where the prior is diffuse, we take limits of the above as $\sigma \to \infty$

$$\lim_{\sigma\to\infty} \mathbb{E}[\mathrm{D} \mid \Psi_R = \psi_R] = \psi_R \tag{9}$$

$$\lim_{\sigma\to\infty} \mathbb{V}[\mathrm{D} \mid \Psi_R = \psi_R] = \sigma_R^2 = \frac{1}{A_R} \tag{10}$$

$$\lim_{\sigma\to\infty} \mathbb{E}[\mathrm{D} \mid \Psi_R = \psi_R, \Psi_S = \psi_S] = \frac{\left(\sigma_S^2 - \rho\sigma_S\sigma_R\right)\psi_R + \left(\sigma_R^2 - \rho\sigma_S\sigma_R\right)\psi_S}{\sigma_S^2 + \sigma_R^2 - 2\rho\sigma_S\sigma_R} \tag{11}$$

$$\lim_{\sigma\to\infty} \mathbb{V}[\mathrm{D} \mid \Psi_R = \psi_R, \Psi_S = \psi_S] = \frac{(1-\rho^2)\sigma_R^2\sigma_S^2}{(\sigma_R^2 + \sigma_S^2 - 2\rho\sigma_S\sigma_R)} = \frac{(1-\rho^2)}{(A_R + A_S - 2\rho\sqrt{A_R A_S})} \tag{12}$$

2 Non-collaborative Model

Proof of Lemma 1. Define $\mathrm{D}_R \equiv \mathrm{D}|\psi_R$. At Stage 2, the retailer solves $\max_Q \pi_R^{non}(Q; A_R, \psi_R)$ where

$$\pi_R^{non}(Q; A_R, \psi_R) = \mathbb{E}_{\mathrm{D}_R}\left[r\min(Q, \mathrm{D}_R) - wQ\right] - \kappa A_r^q \tag{13}$$

$$= \mathbb{E}_{\mathrm{D}_R}[rQ - r(Q - \mathrm{D}_R)^+ - wQ] - \kappa A_r^q \tag{14}$$

$$= (r-w)Q - r\mathbb{E}[(Q - \mathrm{D}_R)^+] - \kappa A_r^q \tag{15}$$

The first order condition is given by $P(\mathrm{D}_R \leq Q) = 1 - \frac{w}{r}$ implying $Q^{non}(\psi) = \mathbb{E}[\mathrm{D}_R] + \sqrt{\mathbb{V}[\mathrm{D}_R]}\varphi_R$ where $\varphi_R = \Phi^{-1}\left(1 - \frac{w}{r}\right)$. The second-order condition confirms that the solution is unique. Therefore,

$$\pi_R^{non}(Q^{non}; A_R, \psi_R) = (r-w)Q^{non} - r\mathbb{E}_{\mathrm{D}_R}[(Q^{non} - \mathrm{D}_R)^+] \tag{16}$$

$$= (r-w)Q^{non} - rQ^{non} + r\mathbb{E}[\mathrm{D}_R] - r\mathbb{E}[(\mathrm{D}_R - Q^{non})^+] \tag{17}$$

$$= (r-w)\mathbb{E}[\mathrm{D}_R] - w\sqrt{\mathbb{V}[\mathrm{D}_R]}\varphi_R - r\int_{Q^n}^{\infty}(x - Q^{non})f_{\mathrm{D}_R}(x)dx \tag{18}$$

By $\int_{Q^{non}}^{\infty}(x - Q^{non})f_{\mathrm{D}_R}(x)dx = \sqrt{\mathbb{V}[\mathrm{D}_R]}\int_{\varphi_R}^{\infty}(x - \varphi_R)\phi(x)dx = \sqrt{\mathbb{V}[\mathrm{D}_R]}[\varphi_R\Phi(\varphi_R) + \phi(\varphi_R) - \varphi_R]$, and observing that $\Phi(\varphi_R) = 1 - \frac{w}{r}$, we can rewrite the profit as

$$= (r-w)\mathbb{E}[\mathrm{D}_R] - w\sqrt{\mathbb{V}[\mathrm{D}_R]}\varphi_R - r\sqrt{\mathbb{V}[\mathrm{D}_R]}\left[-\frac{w}{r}\varphi_R + \phi(\varphi_R)\right] \tag{19}$$

$$= (r-w)\mathbb{E}[\mathrm{D}_R] - r\phi(\varphi_R)\sqrt{\mathbb{V}[\mathrm{D}_R]} \tag{20}$$

$$= (r-w)\psi_R - \frac{x_R}{\sqrt{A_R}}. \tag{21}$$

In Stage 1, the retailer selects accuracy, A_R, to maximize $\Pi_R^{non}(A_R)$, given by

$$\Pi_R^{non}(A_R) = (r-w)\mu - \frac{x_R}{\sqrt{A_R}} - \kappa A_R^q. \tag{22}$$

Solving the first-order condition gives $A_R^{non} = \left(\frac{x_R}{2q\kappa}\right)^{\frac{2}{1+2q}}$. Checking the second-order condition, $\frac{\partial^2 \Pi_R^{non}(A_R)}{\partial A_R^2}\Big|_{A_R=A_R^n} = -\frac{1+2q}{4} x_R \left(\frac{2\kappa q}{x_R}\right)^{\frac{5}{1+2q}} < 0$. For $q \geq 1$, the profit function is concave in A. When $q < 1$, the function is initially concave and unimodal, then convex and decreasing. The inflection point is given by $\left(\frac{3x_R}{4(1-q)q\kappa}\right)^{\frac{2}{1+2q}}$ which is greater than A_R^{non}. □

3 Collaborative Model

Proof of Lemma 2. Define $\mathrm{D}_J \equiv \mathrm{D}|\psi_S, \psi_R$. At Stage 2, the central decision maker solves $\max_Q \pi_{SC}(Q; A_R, A_S, \psi_R, \psi_S)$ where

$$\pi_{SC}(Q; A_R, A_S, \psi_R, \psi_S) = \mathbb{E}_{\mathrm{D}_J}\left[r \min(Q, \mathrm{D}_J) - cQ\right]. \tag{23}$$

Following steps similar to the proof of Lemma 1, the optimal order quantity is given by

$$Q^{col}(\psi_S, \psi_R) = \mathbb{E}[\mathrm{D}_J] + \sqrt{\mathbb{V}[\mathrm{D}_J]}\varphi_J \tag{24}$$

where $\varphi_J = \Phi^{-1}\left(1 - \frac{c}{r}\right)$ and the expected centralized supply chain profit is given by

$$\pi_{SC}(Q^{col}; A_R, A_S, \psi_R, \psi_S) = (r-c)\mathbb{E}[\mathrm{D}_J] - r\phi(\varphi_J)\sqrt{\mathbb{V}[\mathrm{D}_J]}. \tag{25}$$

In Stage 1, the central decision maker solves $\max_{A_S, A_R} \Pi_{SC}(A_R, A_S)$ where

$$\Pi_{SC}(A_R, A_S) = \mathbb{E}_{\Psi_R, \Psi_S}\left[\pi_{SC}(Q^{col}; A_R, A_S, \psi_R, \psi_S)\right] - \kappa A_R^q - \kappa A_S^q \tag{26}$$

$$= (r-c)\mu - x_J \sqrt{\frac{1-\rho^2}{(A_R + A_S - 2\rho\sqrt{A_R A_S})}} - \kappa A_R^q - \kappa A_S^q. \tag{27}$$

For a given $\hat{A}$, define $\hat{A}_R = \hat{A} - \delta$ and $\hat{A}_S = (2\hat{A}^q - (\hat{A} - \delta)^q)^{1/q}$. Then, total forecasting cost, $\kappa \hat{A}_R^q + \kappa \hat{A}_S^q$, is constant for all $\delta < \hat{A}$ and equals $2\kappa \hat{A}^q$. Thus, δ defines all pairs of accuracies that can be obtained at the same cost. Substituting into the variance of D_J reveals that variance is quasiconcave: everywhere nonincreasing, everywhere nondecreasing, or increasing than decreasing, in δ. Thus, two candidate solutions exist: a symmetric solution satisfying $A_R = A_S$ or a corner solution satisfying the constraint $A_R = \rho^2 A_S$ in which only one signal is informative.

JOINT FORECASTING The first-order conditions are given implicitly by:

$$A_S^{1-q}\left(\sqrt{A_RA_S}-\rho A_R\right)=\frac{2\kappa q(A_R+A_S-2\rho\sqrt{A_RA_S})^{3/2}\sqrt{A_RA_S}}{x_J\sqrt{1-\rho^2}} \tag{28}$$

$$A_R^{1-q}\left(\sqrt{A_RA_S}-\rho A_S\right)=\frac{2\kappa q(A_R+A_S-2\rho\sqrt{A_RA_S})^{3/2}\sqrt{A_RA_S}}{x_J\sqrt{1-\rho^2}} \tag{29}$$

When $A_R = A_S$, the above reduce to $A_R = A_S = \left(\frac{x_J}{2q\kappa}\left(\frac{\sqrt{1+\rho}}{2\sqrt{2}}\right)\right)^{\frac{2}{1+2q}}$.

TARGETED FORECASTING Evaluating Π_{SC} when $A_S = \rho^2 A_R$ yields $(r-c)\mu - x_J A_R^{-1/2} - (1+\rho^{2q})\kappa A_R^q$, which obtains a maximum at $A_R = \left(\frac{x_J}{2q\kappa}\left(\frac{1}{1+\rho^{2q}}\right)\right)^{\frac{2}{1+2q}}$ and by construction $A_S = \rho^2 A_R$.

OPTIMAL SOLUTION Substituting the joint and targeted forecasting solutions into Π_{SC} yields:

$$\Pi_{SC}^{joint} = (r-c)\mu - 2^{\frac{1-q}{1+2q}}(1+2q)\left(\frac{x_J\sqrt{1+\rho}}{2\kappa q}\right)^{\frac{2q}{1+2q}}\kappa \tag{30}$$

$$\Pi_{SC}^{targeted} = (r-c)\mu - (1+2q)\left(1+\rho^{2q}\right)^{\frac{1}{1+2q}}\left(\frac{x_J}{2\kappa q}\right)^{\frac{2q}{1+2q}}\kappa \tag{31}$$

Joint forecasting is optimal if

$$\Pi_{SC}^{joint} > \Pi_{SC}^{targeted} \tag{32}$$

$$\equiv \quad 2^{\frac{1-q}{1+2q}}(1+2q)\left(\frac{x_J\sqrt{1+\rho}}{2\kappa q}\right)^{\frac{2q}{1+2q}}\kappa < (1+2q)\left(1+\rho^{2q}\right)^{\frac{1}{1+2q}}\left(\frac{x_J}{2\kappa q}\right)^{\frac{2q}{1+2q}}\kappa \tag{33}$$

$$\equiv \quad 2^{\frac{1-q}{1+2q}}(1+\rho)^{\frac{q}{1+2q}} < \left(1+\rho^{2q}\right)^{\frac{1}{1+2q}} \tag{34}$$

$$\equiv \quad 2^{1-q}(1+\rho)^q < \left(1+\rho^{2q}\right). \qquad \square$$

In the collaborative model, joint forecasting is optimal precisely when the resulting demand forecast is higher under the joint forecasting solution than the targeted forecasting solution.

4 Comparison of the Non-Collaborative and Collaborative Models

Proof of Proposition 1. From Lemma 1 and 2, the condition $A_R^{non} > A_F^{col}$ is satisfied when

$$\left(\frac{x_R}{2q\kappa}\right)^{\frac{2}{1+2q}} > \max\left\{\frac{2}{1+\rho}\left(\frac{x_J}{2q\kappa}\left(\frac{\sqrt{1+\rho}}{2\sqrt{2}}\right)\right)^{\frac{2}{1+2q}}, \left(\frac{x_J}{2q\kappa}\left(\frac{1}{1+\rho^{2q}}\right)\right)^{\frac{2}{1+2q}}\right\} \tag{35}$$

where the first term on the right-hand side is equal to the final accuracy of the demand forecast when the optimal is given by joint forecasting and the second term is the final demand accuracy when the optimal is given by targeted forecasting. Simplifying the above condition we get,

$$\frac{x_J}{x_R} = \frac{\phi\left(\Phi^{-1}\left(1-\frac{c}{r}\right)\right)}{\phi\left(\Phi^{-1}\left(1-\frac{w}{r}\right)\right)} < \min\{2^{1-q}(1+\rho)^q, (1+\rho^{2q})\}. \qquad \square$$

Proof of Corollary 1.1. (i). When the optimal solution is given by targeted forecasting, $A_R^{non} > A_F^{col}$ requires $\frac{x_J}{x_R} < 1+\rho^{2q}$. Since $1+\rho^{2q} \geq 1$, the condition in Proposition 1 is satisfied for any ρ, q if $\frac{x_J}{x_R} < 1$.

$$x_J < x_R \tag{36}$$

$$\equiv \quad r\phi\left(\Phi^{-1}\left(1-\frac{c}{r}\right)\right) < r\phi\left(\Phi^{-1}\left(1-\frac{w}{r}\right)\right) \tag{37}$$

$$\equiv \quad \left|\Phi^{-1}\left(1-\frac{c}{r}\right)\right| > \left|\Phi^{-1}\left(1-\frac{w}{r}\right)\right| \tag{38}$$

$$\equiv \quad \left|\frac{1}{2}-\frac{c}{r}\right| > \left|\frac{1}{2}-\frac{w}{r}\right| \tag{39}$$

$$\equiv \quad |r-2c| > |r-2w| \tag{40}$$

When $r \geq 2w$, both terms inside the absolute values are nonnegative, so the inequality holds since $w > c$. Assume $2w > r > w + c$. Then

$$|r-2c| = r-2c > r-2(r-w) = 2w-r = |r-2w|. \qquad \square$$

Thus, $x_J < x_R$ if $w + c < r$ which is equivalent to the condition in the corollary.

(ii). When the optimal solution is given by joint forecasting, $A_R^{non} > A_F^{col}$ requires $\frac{x_J}{x_R} <$

$\beta \equiv 2^{1-q}(1+\rho)^q$. This is equivalent to

$$x_J < \beta x_R \tag{41}$$

$$\equiv \quad r\phi\left(\Phi^{-1}\left(1-\frac{c}{r}\right)\right) < \beta r\phi\left(\Phi^{-1}\left(1-\frac{w}{r}\right)\right) \tag{42}$$

$$\equiv \quad \left(\Phi^{-1}\left(1-\frac{c}{r}\right)\right)^2 > \left(\Phi^{-1}\left(1-\frac{w}{r}\right)\right)^2 - 2\log\beta \tag{43}$$

Using the approximation of the inverse error function given in [11],

$$\approxeq \quad \log\left(\left(1-\frac{c}{r}\right)\frac{c}{r}\right) < \log\left(\left(1-\frac{w}{r}\right)\frac{w}{r}\right) + \log\beta \tag{44}$$

$$\equiv \quad \left(1-\frac{c}{r}\right)\frac{c}{r} < \beta\left(1-\frac{w}{r}\right)\frac{w}{r} \tag{45}$$

(iii) Note that $1+\rho^{2q} \leq 2$, implying that $\min\{2^{1-q}(1+\rho)^q, (1+\rho^{2q})\} \leq 2$. Thus, it is sufficient to show that $\frac{x_J}{x_R} > 2$, which is the inverse of the condition from part (ii) for $\beta = 2$ Thus,

$$x_J > 2x_R \tag{46}$$

$$\approxeq \quad \left(1-\frac{c}{r}\right)\frac{c}{r} > 2\left(1-\frac{w}{r}\right)\frac{w}{r}. \tag{47}$$

Rearranging the above condition yields $\frac{w}{r} + \frac{c}{r} \gtrapprox 1 + \frac{wc}{r(2w-c)}$.